CHAPTER 1
ELECTRIC FIELDS

1.1 *Introduction*

This is the first in a series of chapters on electricity and magnetism. Much of it will be aimed at an introductory level suitable for first or second year students, or perhaps some parts may also be useful at high school level. Occasionally, as I feel inclined, I shall go a little bit further than an introductory level, though the text will not be enough for anyone pursuing electricity and magnetism in a third or fourth year honours class. On the other hand, students embarking on such advanced classes will be well advised to know and understand the contents of these more elementary notes before they begin.

The subject of electromagnetism is an amalgamation of what were originally studies of three apparently entirely unrelated phenomena, namely *electrostatic* phenomena of the type demonstrated with pieces of amber, pith balls, and ancient devices such as Leyden jars and Wimshurst machines; magnetism, and the phenomena associated with lodestones, compass needles and Earth's magnetic field; and current electricity – the sort of electricity generated by chemical cells such as Daniel and Leclanché cells. These must have seemed at one time to be entirely different phenomena. It wasn't until 1820 that Oersted discovered (during the course of a university lecture, so the story goes) that an electric current is surrounded by a magnetic field, which could deflect a compass needle. The several phenomena relating the apparently separate phenomena were discovered during the nineteenth century by scientists whose names are immortalized in many of the units used in electromagnetism – Ampère, Ohm, Henry, and, especially, Faraday. The basic phenomena and the connections between the three disciplines were ultimately described by Maxwell towards the end of the nineteenth century in four famous equations. This is not a history book, and I am not qualified to write one, but I strongly commend to anyone interested in the history of physics to learn about the history of the growth of our understanding of electromagnetic phenomena, from Gilbert's description of terrestrial magnetism in the reign of Queen Elizabeth I, through Oersted's discovery mentioned above, up to the culmination of Maxwell's equations.

This set of notes will be concerned primarily with a description of electricity and magnetism as natural phenomena, and it will be treated from the point of view of a "pure" scientist. It will *not* deal with the countless electrical devices that we use in our everyday life – how they work, how they are designed and how they are constructed. These matters are for electrical and electronics engineers. So, you might ask, if your primary interest in electricity is to understand how machines, instruments and electrical equipment work, is there any point in studying electricity from the very "academic" and abstract approach that will be used in these notes, completely divorced as they appear to be from the world of practical reality? The answer is that electrical engineers *more than anybody* must understand the basic scientific principles before they even begin to apply them to the design of practical appliances. So – do not even think of electrical engineering until you have a thorough understanding of the basic scientific principles of the subject.

This chapter deals with the basic phenomena, definitions and equations concerning *electric fields.*

1.2 *Triboelectric Effect*

In an introductory course, the basic phenomena of electrostatics are often demonstrated with "pith balls" and with a "gold-leaf electroscope". A *pith ball* used to be a small, light wad of pith extracted from the twig of an elder bush, suspended by a silk thread. Today, it is more likely to be either a ping-pong ball, or a ball of styrofoam, suspended by a nylon thread – but, for want of a better word, I'll still call it a pith ball. I'll describe the gold-leaf electroscope a little later.

It was long ago noticed that if a sample of amber (fossilized pine sap) is rubbed with cloth, the amber became endowed with certain apparently wonderful properties. For example, the amber would be able to attract small particles of fluff to itself. The effect is called the *triboelectric effect.* [Greek τρίβος (rubbing) + ήλεκτρον (amber)] The amber, after having been rubbed with cloth, is said to bear an *electric charge*, and space in the vicinity of the charged amber within which the amber can exert its attractive properties is called an *electric field.*

Amber is by no means the best material to demonstrate triboelectricity. Modern plastics (such as a comb rubbed through the hair) become easily charged with electricity (provided that the plastic, the cloth or the hair, and the atmosphere, are *dry*). Glass rubbed with silk also carries an electric charge – but, as we shall see in the next section, the charge on glass rubbed with silk seems to be not quite the same as the charge on plastic rubbed with cloth.

1.3 *Experiments with Pith Balls*

A pith ball hangs vertically by a thread. A plastic rod is charged by rubbing with cloth. The charged rod is *brought close to the pith ball without touching it.* It is observed that the charged rod weakly *attracts* the pith ball. This may be surprising – and you are right to be surprised, for the pith ball carries no charge. For the time being we are going to put this observation to the back of our minds, and we shall defer an explanation to a later chapter. Until then it will remain a small but insistent little puzzle.

We now *touch* the pith ball with the charged plastic rod. Immediately, some of the magical property (i.e. some of the electric charge) of the rod is transferred to the pith ball, and we observe that thereafter the ball is strongly *repelled* from the rod. We conclude that two electric charges repel each other. Let us refer to the pith ball that we have just charged as Ball A.

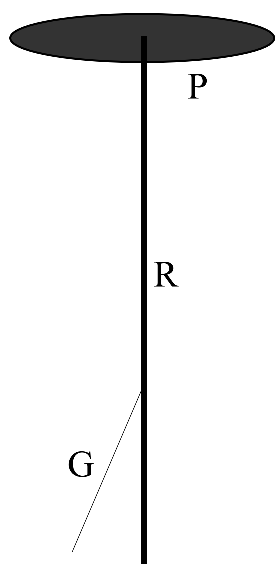
Now let's do exactly the same experiment with the glass rod that has been rubbed with silk. We bring the charged glass rod close to an uncharged Ball B. It initially attracts it weakly – but we'll have to wait until Chapter 2 for an explanation of this unexpected behaviour. However, as soon as we *touch* Ball B with the glass rod, some charge is transferred to the ball, and the rod thereafter repels it. So far, no obvious difference between the properties of the plastic and glass rods.

But... now bring the *glass* rod close to *Ball A*, and we see that Ball A is strongly *attracted*. And if we bring the *plastic* rod close to *Ball B*, it, too, is strongly *attracted.* Furthermore, Balls A and B *attract each other.*

We conclude that there are *two kinds* of electric charge, with exactly opposite properties. We arbitrarily call the kind of charge on the glass rod and on Ball B *positive* and the charge on the plastic rod and Ball A *negative*. We observe, then, that *like* charges (i.e. those of the *same* sign) *repel* each other, and *unlike* charges (i.e. those of *opposite* sign) *attract* each other.

1.4 *Experiments with a Gold-leaf Electroscope*

FIGURE I.1



A gold-leaf electroscope has a vertical rod R attached to a flat metal plate P. Gold is a malleable metal which can be hammered into extremely thin and light sheets. A light gold leaf G is attached to the lower end of the rod.

If the electroscope is positively charged by touching the plate with a positively charged glass rod, G will be repelled from R, because both now carry a positive charge.

You can now experiment as follows. Bring a positively charged glass rod close to P. The leaf G diverges further from R. We now know that this is because the metal (of which P, R and G are all composed) contains *electrons*, which are negatively charged particles that can move about more or less freely inside the metal. The approach of the positively charged glass rod to P attracts electrons towards P, thus increasing the excess positive charge on G and the bottom end of R. G therefore moves away from R.

If on the other hand you were to approach P with a negatively charged plastic rod, electrons would be repelled from P down towards the bottom of the rod, thus reducing the excess positive charge there. G therefore approaches R.

Now try another experiment. Start with the electroscope uncharged, with the gold leaf hanging limply down. (This can be achieved by touching P briefly with your finger.) Approach P with a negatively charged plastic rod, but don't touch. The gold leaf diverges from R. Now, briefly touch P with a finger of your free hand. Negatively charged electrons run down through your body to ground (or earth). Don't worry – you won't feel a thing. The gold leaf collapses, though by this time the electroscope bears a positive charge, because it has lost some electrons through your body. Now remove the plastic rod. The gold leaf diverges again. By means of the negatively charged plastic rod and some deft work with your finger, you have induced a positive charge on the electroscope. You can verify this by approaching P alternately with a plastic (negative) or glass (positive) rod, and watch what happens to the gold leaf.

1.5 *Coulomb's Law*

If you are interested in the history of physics, it is well worth reading about the important experiments of Charles Coulomb in 1785. In these experiments he had a small fixed metal sphere which he could charge with electricity, and a second metal sphere attached to a vane suspended from a fine torsion thread. The two spheres were charged and, because of the repulsive force between them, the vane twisted round at the end of the torsion thread. By this means he was able to measure precisely the small forces between the charges, and to determine how the force varied with the amount of charge and the distance between them.

From these experiments resulted what is now known as *Coulomb's Law*. Two electric charges of like sign repel each other with a force that is proportional to the product of their charges and inversely proportional to the square of the distance between them:

$$F \propto \frac{Q_1 Q_2}{r^2}. \qquad 1.5.1$$

Here Q_1 and Q_2 are the two charges and r is the distance between them.

We could in principle use any symbol we like for the constant of proportionality, but in standard SI (Système International) practice, the constant of proportionality is written as $\frac{1}{4\pi\epsilon}$, so that Coulomb's Law takes the form

$$F = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{r^2}. \qquad 1.5.2$$

Here ε is called the *permittivity* of the medium in which the charges are situated, and it varies from medium to medium. The permittivity of a *vacuum* (or of "free space") is given the symbol ε_0. Media other than a vacuum have permittivities a little greater than ε_0. The permittivity of air is very little different from that of free space, and, unless specified otherwise, I shall assume that all experiments described in this chapter are done either in free space or in air, so that I shall write Coulomb's Law as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{r^2}. \qquad 1.5.3$$

You may wonder – why the factor 4π? In fact it is very convenient to define the permittivity in this manner, with 4π in the denominator, because, as we shall see, it will ensure that all formulas that describe situations of spherical symmetry will include a 4π, formulas that describe situations of cylindrical symmetry will include 2π, and no π will appear in formulas involving uniform fields. Some writers (particularly those who favour cgs units) prefer to incorporate the 4π into the definition of the permittivity, so that Coulomb's law appears in the form $F = Q_1Q_2/(\varepsilon_0 r^2)$, though it is standard SI practice to define the permittivity as in equation 1.5.3. The permittivity defined by equation 1.5.3 is known as the "rationalized" definition of the permittivity, and it results in much simpler formulas throughout electromagnetic theory than the "unrationalized" definition.

The SI unit of charge is the *coulomb*, C. Unfortunately at this stage I cannot give you an exact definition of the coulomb, although, if a current of 1 amp flows for a second, the amount of electric charge that has flowed is 1 coulomb. This may at first seem to be very clear, until you reflect that we have not yet defined what is meant by an amp, and that, I'm afraid, will have to come in a much later chapter.

Until then, I can give you some small indications. For example, the charge on an electron is about -1.6022×10^{-19} C, and the charge on a proton is about $+1.6022 \times 10^{-19}$ C. That is to say, a collection of 6.24×10^{18} protons, if you could somehow bundle them all together and stop them from flying apart, amounts to a charge of 1 C. A *mole* of protons (i.e. 6.022×10^{23} protons) which would have a mass of about one gram, would have a charge of 9.65×10^4 C, which is also called a *faraday* (which is *not at all* the same thing as a *farad*).

The charges involved in our experiments with pith balls, glass rods and gold-leaf electroscopes are very small in terms of coulombs, and are typically of the order of nanocoulombs.

The permittivity of free space has the approximate value

$$\varepsilon_0 = 8.8542 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}.$$

Later on, when we know what is meant by a "farad", we shall use the units F m^{-1} to describe permittivity – but that will have to wait until section 5.2.

You may well ask how the permittivity of free space is measured. A brief answer might be "by carrying out experiments similar to those of Coulomb". However – and this is rather a long story, which I shall not describe here – it turns out that since we today define the metre by *defining* the speed of light, *c*, to be exactly 2.997 925 58 × 10^8 m s^{-1}, the permittivity of free space has a *defined value*, given, in SI units, by

$$4\pi\varepsilon_0 = \frac{10^7}{c^2}.$$

It is therefore not necessary to *measure* ε_0 any more than it is necessary to *measure c*. But that, as I say, is a long story.

From the point of view of *dimensional analysis*, electric charge cannot be expressed in terms of M, L and T, but it has a dimension, Q, of its own. (This assertion is challenged by some, but this is not the place to discuss the reasons. I may add a chapter, eventually, discussing this point much later on.) We say that the dimensions of electric charge are Q.

Exercise: Show that the dimensions of permittivity are

$$[\varepsilon_0] = \mathrm{M}^{-1}\,\mathrm{L}^{-3}\,\mathrm{T}^2\,\mathrm{Q}^2.$$

I shall strongly advise the reader to work out and make a note of the dimensions of every new electric or magnetic quantity as it is introduced.

Exercise: Calculate the magnitude of the force between two point charges of 1 C each (that's an enormous charge!) 1 m apart *in vacuo*.

The answer, of course, is $1/(4\pi\varepsilon_0)$, and that, as we have just seen, is $c^2/10^7 = 9 \times 10^9$ N, which is equal to the weight of a mass of 9.2×10^5 tonnes or nearly a million tonnes.

Exercise: Calculate the ratio of the electrostatic to the gravitational force between two electrons. The numbers you will need are: $Q = 1.60 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg, $\varepsilon_0 = 8.85 \times 10^{-12}$ N m^2 C^{-2}, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} .

The answer, which is independent of their distance apart, since both forces fall off inversely as the square of the distance, is $Q^2/(4\pi\varepsilon_0 Gm^2)$, (and you should verify that this is dimensionless), and this comes to 4.2×10^{42}. This is the basis of the oft-heard statement that electrical forces are 10^{42} times as strong as gravitational forces – but such a statement out of context is rather meaningless. For example, the gravitational force between Earth and Moon is much more than the electrostatic force (if any) between them, and cosmologists could make a good case for saying that the strongest forces in the Universe are gravitational.

The ratio of the permittivity of an insulating substance to the permittivity of free space is its *relative permittivity*, also called its *dielectric constant.* The dielectric constants of many commonly-encountered insulating substances are of order "a few". That is, somewhere between 2 and 10. Pure water has a dielectric constant of about 80, which is quite high (but bear in mind that most water is far from pure and is not an insulator.) Some special substances, known as *ferroelectric* substances, such as strontium titanate $SrTiO_3$, have dielectric constants of a few hundred.

1.6 *Electric Field* ***E***

The region around a charged body within which it can exert its electrostatic influence may be called an *electric field*. In principle, it extends to infinity, but in practice it falls off more or less rapidly with distance. We can define the *intensity* or *strength* E of an electric field as follows. Suppose that we place a small test charge q in an electric field. This charge will then experience a force. The ratio of the force to the charge is called the *intensity of the electric field*, or, more usually, simply the *electric field*. Thus I have used the words "electric field" to mean either the region of space around a charged body, or, quantitatively, to mean its intensity. Usually it is clear from the context which is meant, but, if you wish, you may elect to use the longer phrase "intensity of the electric field" if you want to remove all doubt. The field and the force are in the same direction, and the electric field is a vector quantity, so the definition of the electric field can be written as

$$\mathbf{F} = Q\mathbf{E} \, . \qquad 1.6.1$$

The SI units of electric field are newtons per coulomb, or N C^{-1}. A little later, however, we shall come across a unit called a *volt*, and shall learn that an alternative (and more usual) unit for electric field is volts per metre, or V m^{-1}. The dimensions are $MLT^{-2}Q^{-1}$.

You may have noticed that I supposed that we place a "small" test charge in the field, and you may have wondered why it had to be small, and how small. The problem is that, if we place a large charge in an electric field, this will change the configuration of the electric field and hence frustrate our efforts to measure it accurately. So – it has to be sufficiently small so as not to change the configuration of the field that we are trying to measure. How small is that? Well, it will have to mean infinitesimally small. I hope that is clear! (It is a bit like that pesky particle of negligible mass m that keeps appearing in mechanics problems!)

We now need to calculate the intensity of an electric field in the vicinity of various shapes and sizes of charged bodes, such as rods, discs, spheres, and so on.

1.6.1 *Field of a point charge*

It follows from equation 1.5.3 and the definition of electric field intensity that the electric field at a distance r from a point charge Q is of magnitude

$$E = \frac{Q}{4\pi\epsilon_0 r^2} . \qquad 1.6.2$$

This can be written in vector form:

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r} . \qquad 1.6.3$$

Here $\hat{\mathbf{r}}$ is a unit vector in the radial direction, and $\mathbf{r}$ is a vector of length r in the radial direction.

1.6.2 *Spherical Charge Distributions*

I shall not here give calculus derivations of the expressions for electric fields resulting from spherical charge distributions, since they are identical with the derivations for the gravitational fields of spherical mass distributions in the Classical Mechanics "book" of these physics notes, provided that you replace mass by charge and G by $-1/(4\pi\epsilon_0)$. See Chapter 5, subsections 5.4.8 and 5.4.9 of Celestial Mechanics. Also, we shall see later that they can be derived more easily from Gauss's law than by calculus. I shall, however, give the *results* here.

At a distance r from the centre of a *hollow spherical shell* of radius a bearing a charge Q, the electric field is *zero* at any point *inside* the sphere (i.e. for $r < a$). For a point *outside* the sphere (i.e. $r > a$) the field intensity is

$$E = \frac{Q}{4\pi\epsilon_0 r^2} . \qquad 1.6.4$$

This is the same as if all the charge were concentrated at a point at the centre of the sphere.

If you have a *spherically-symmetric distribution of charge Q contained within a spherical volume of radius a,* this can be considered as a collection of nested hollow spheres. It follows that at a point *outside* a spherically-symmetric distribution of charge, the field at a distance r from the centre is again

$$E = \frac{Q}{4\pi\epsilon_0 r^2} . \qquad 1.6.5$$

That is, it is the same as if all the charge were concentrated at the centre. However, at a point *inside* the sphere, the charge beyond the distance r from the centre contributes zero to the electric field; the electric field at a distance r from the centre is therefore just

$$E = \frac{Q_r}{4\pi\epsilon_0 r^2}. \qquad 1.6.6$$

Here Q_r is the charge within a radius r. If the charge is uniformly distributed throughout the sphere, this is related to the total charge by $Q_r = \left(\frac{r}{a}\right)^3 Q$, where Q is the total charge. Therefore, for a uniform spherical charge distribution the field inside the sphere is

$$E = \frac{Qr}{4\pi\epsilon_0 a^3}. \qquad 1.6.7$$

That is to say, it increases linearly from centre to the surface, where it reaches a value of $\frac{Q}{4\pi\epsilon_0 a^2}$, whereafter it decreases according to equation 1.6.5.

It is not difficult to imagine some electric charge distributed (uniformly or otherwise) throughout a finite spherical volume, but, because like charges repel each other, it may not be easy to realize this idealized situation in practice. In particular, if a *metal* sphere is charged, since charge can flow freely through a metal, the self-repulsion of charges will result in all the charge residing on the *surface* of the sphere, which then behaves as a hollow spherical charge distribution with zero electric field within.

1.6.3 *A Long, Charged Rod*

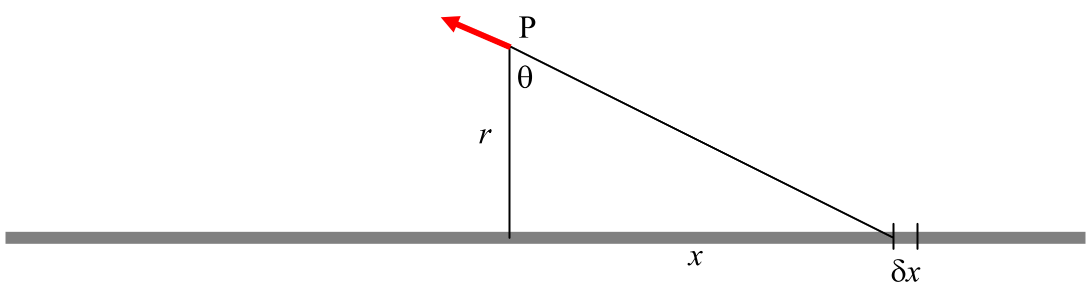


FIGURE I.1

A long rod bears a charge of λ coulombs per metre of its length. What is the strength of the electric field at a point P at a distance r from the rod?

Consider an element δx of the rod at a distance $(r^2 + x^2)^{1/2}$ from the rod. It bears a charge $\lambda\,\delta x$. The contribution to the electric field at P from this element is

$\dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda\,\delta x}{r^2 + x^2}$ in the direction shown. The radial component of this is $\dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{\lambda\,\delta x}{r^2 + x^2}\cos\theta.$ But $x = r\tan\theta,\ \ \delta x = r\sec^2\theta\,\delta\theta$ and $r^2 + x^2 = r^2\sec^2\theta.$ Therefore the radial component of the field from the element δx is $\dfrac{\lambda}{4\pi\epsilon_0 r}\cos\theta\,\delta\theta.$ To find the radial component of the field from the entire rod, we integrate along the length of the rod. If the rod is infinitely long (or if its length is much greater than r), we integrate from $\theta = -\pi/2$ to $+\pi/2$, or, what amounts to the same thing, from 0 to $\pi/2$, and double it. Thus the radial component of the field is

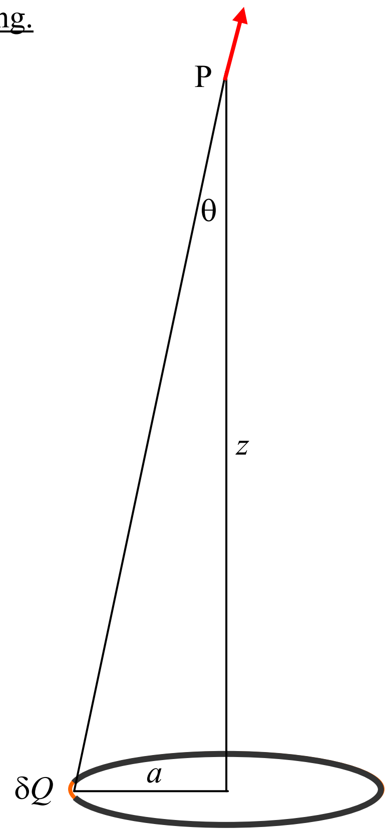
$$E = \frac{2\lambda}{4\pi\epsilon_0 r}\int_0^{\pi/2}\cos\theta\ \delta\theta = \frac{\lambda}{2\pi\epsilon_0 r}\cdot \qquad 1\ 6.8$$

The component of the field parallel to the rod, by considerations of symmetry, is zero, so equation 1.6.8 gives the total field at a distance r from the rod, and it is directed radially away from the rod.

Notice that equation 1.6.4 for a spherical charge distribution has $4\pi r^2$ in the denominator, while equation 1.6.8, dealing with a problem of cylindrical symmetry, has $2\pi r$.

1.6.4 *Field on the Axis of and in the Plane of a Charged Ring*

Field on the axis of a charged ring.



Ring, radius a, charge Q. Field at P from element of charge $\delta Q = \dfrac{\delta Q}{4\pi\epsilon_0 (a^2 + z^2)}$.

Vertical component of this $= \dfrac{\delta Q \cos\theta}{4\pi\epsilon_0 (a^2 + z^2)} = \dfrac{\delta Q z}{4\pi\epsilon_0 (a^2 + z^2)^{3/2}}$.

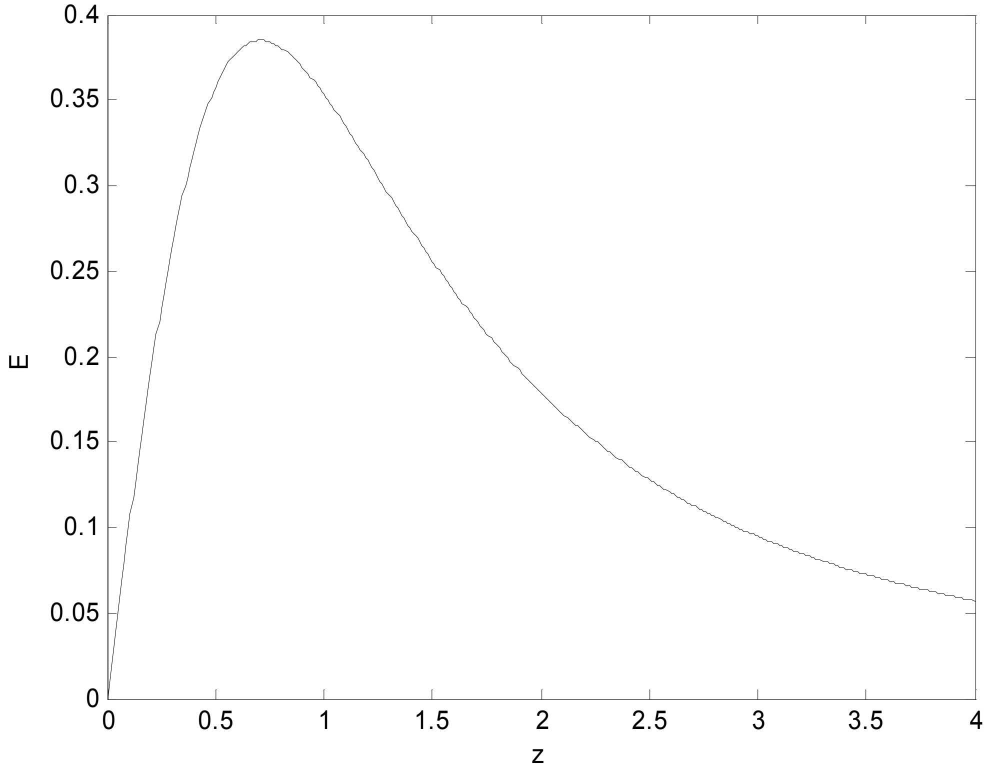
Integrate for entire ring:

$$\text{Field} \quad E = \frac{Q}{4\pi\epsilon_0} \frac{z}{(a^2 + z^2)^{3/2}}.$$

In terms of dimensionless variables:

$$E = \frac{z}{(1 + z^2)^{3/2}},$$

where E is in units of $\frac{Q}{4\pi\varepsilon_0 a^2}$, and z is in units of a.



From calculus, we find that this reaches a maximum value of $\frac{2\sqrt{3}}{9} = \underline{\underline{0.3849}}$ at $z = 1/\sqrt{2} = \underline{\underline{0.7071}}$.

It reaches half of its maximum value where $\frac{z}{(1+z^2)^{3/2}} = \frac{\sqrt{3}}{9}$.

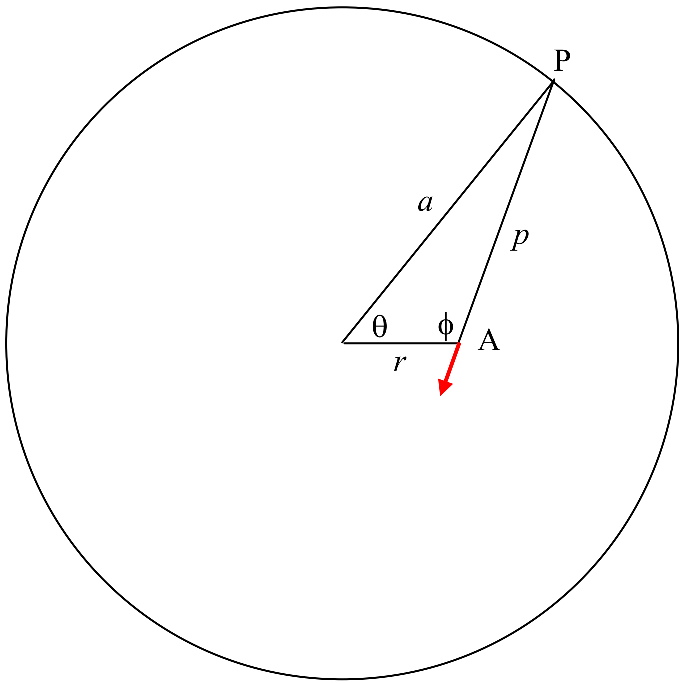
That is, $3 - 72Z + 9Z^2 + 3Z^3 = 0$, where $Z = z^2$.

The two positive solution are $Z = 0.041889$ and 3.596267.

That is, $\underline{\underline{z = 0.2047 \text{ and } 1.8964.}}$

Field in the plane of a charged ring.

We suppose that we have a ring of radius a bearing a charge Q. We shall try to find the field at a point in the plane of the ring and at a distance r $(0 \leq r < a)$ from the centre of the ring.



Consider an element $\delta\theta$ of the ring at P. The charge on it is $\dfrac{Q\delta\theta}{2\pi}$. The field at A due this element of charge is

$$\frac{1}{4\pi\epsilon_0}\cdot\frac{Q\delta\theta}{2\pi}\cdot\frac{1}{a^2+r^2-2ar\cos\theta} = \frac{Q}{4\pi\epsilon_0 . 2\pi a^2}\cdot\frac{\delta\theta}{b-c\cos\theta},$$

where $b = 1 + r^2/a^2$ and $c = 2r/a$. The component of this toward the centre is

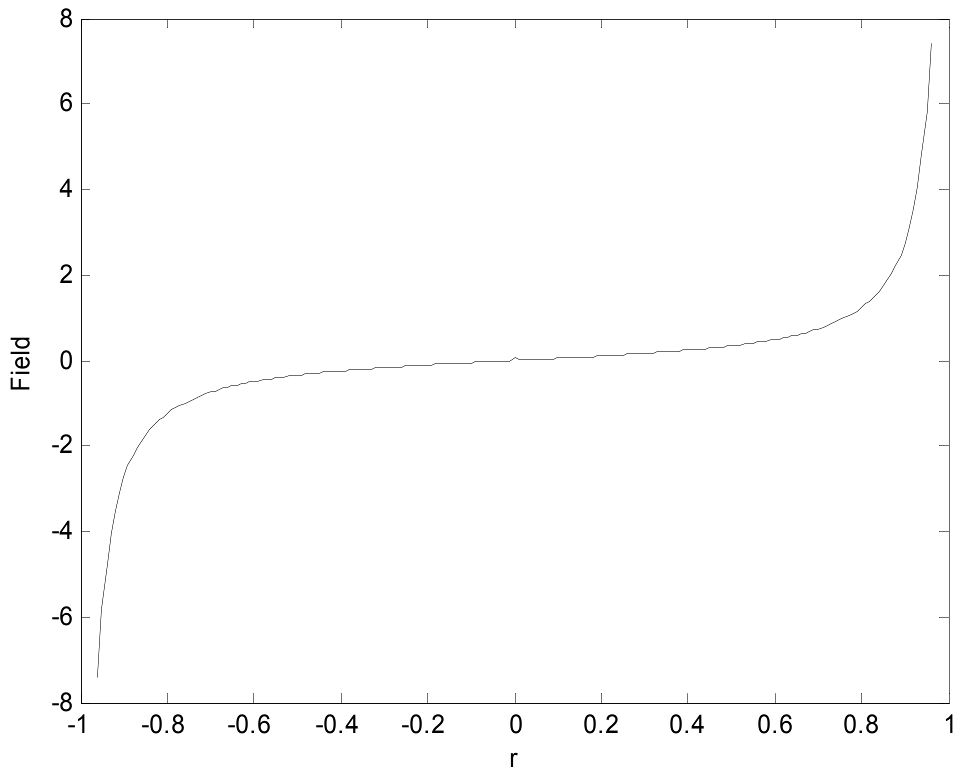
$$-\frac{Q}{4\pi\epsilon_0 . 2\pi a^2}\cdot\frac{\cos\phi\,\delta\theta}{b-c\cos\theta}.$$

To find the field at A due to the entire ring, we must express ϕ in terms of θ, r and a, and integrate with respect to θ from 0 to 2π (or from 0 to π and double it). The necessary relations are

$$p^2 = a^2 + r^2 - 2ar\cos\theta,$$

$$\cos\phi = \frac{r^2 + p^2 - a^2}{2rp}.$$

The result of the numerical integration is shown below, in which the field is expressed in units of $Q/(4\pi\varepsilon_0 a^2)$ and r is in units of a.



1.6.5 *Field on the Axis of a Uniformly Charged Disc*

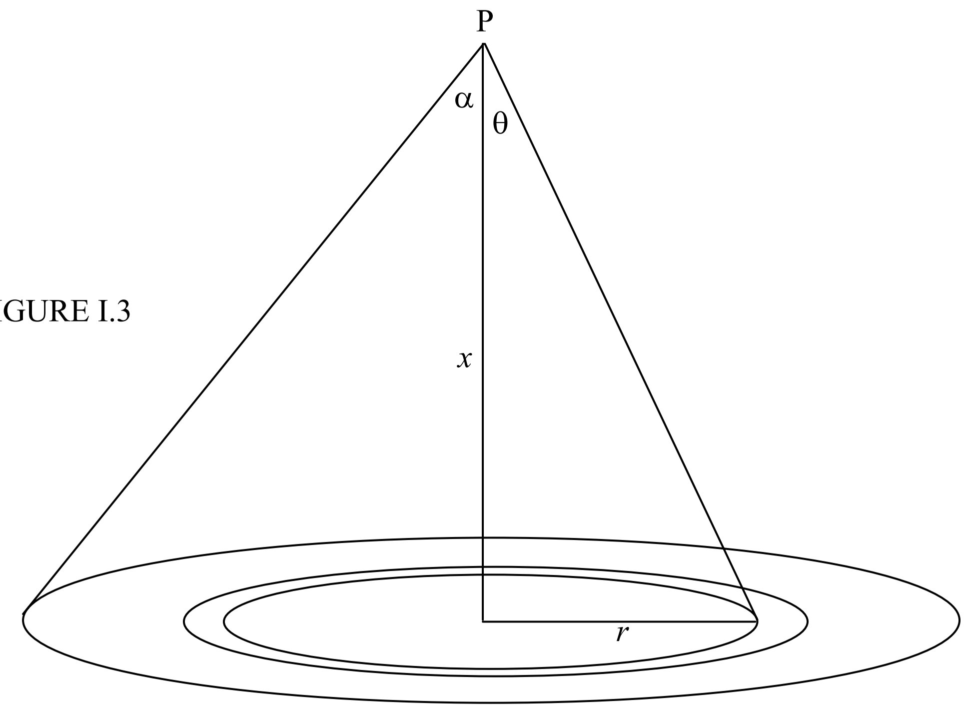


FIGURE I.3

We suppose that we have a circular disc of radius a bearing a surface charge density of σ coulombs per square metre, so that the total charge is $Q = \pi a^2 \sigma$. We wish to calculate the field strength at a point P on the axis of the disc, at a distance x from the centre of the disc.

Consider an elemental annulus of the disc, or radii r and $r + \delta r$. Its area is $2\pi r \delta r$ and so it carries a charge $2\pi\sigma r \delta r$. Using the result of subsection1.6.4, we see that the field at P from this charge is

$$\frac{2\pi\sigma r\,\delta r}{4\pi\epsilon_0} \cdot \frac{x}{(r^2 + x^2)^{3/2}} = \frac{\sigma x}{2\epsilon_0} \cdot \frac{r\,\delta r}{(r^2 + x^2)^{3/2}}.$$

But $r = x\tan\theta$, $\delta r = x\sec^2\theta\,\delta\theta$ and $(r^2 + x^2)^{1/2} = x\sec\theta$. Thus the field from the elemental annulus can be written

$$\frac{\sigma}{2\epsilon_0}\sin\theta\,\delta\theta.$$

The field from the entire disc is found by integrating this from $\theta = 0$ to $\theta = \alpha$ to obtain

$$E = \frac{\sigma}{2\epsilon_0}(1 - \cos\alpha) = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{x}{(a^2 + x^2)^{1/2}}\right). \qquad 1.6.11$$

This falls off monotonically from $\sigma/(2\epsilon_0)$ just above the disc to zero at infinity.

1.6.6 *Field of a Uniformly Charged Infinite Plane Sheet*

All we have to do is to put $\alpha = \pi/2$ in equation 1.6.10 to obtain

$$E = \frac{\sigma}{2\epsilon_0}. \qquad 1.6.12$$

This is independent of the distance of P from the infinite charged sheet. The electric field lines are uniform parallel lines extending to infinity.

Summary

Point charge Q: $\quad E = \dfrac{Q}{4\pi\epsilon_0 r^2}$.

Hollow Spherical Shell: $\quad E$ = zero inside the shell,

$E = \dfrac{Q}{4\pi\epsilon_0 r^2}$ outside the shell.

Infinite charged rod: $\quad E = \dfrac{\lambda}{2\pi\epsilon_0 r}$.

Infinite plane sheet: $\quad E = \dfrac{\sigma}{2\epsilon_0}$.

1.7 *Electric Field* ***D***

We have been assuming that all "experiments" described have been carried out in a vacuum or (which is almost the same thing) in air. But what if the point charge, the infinite rod and the infinite charged sheet of section 1.6 are all immersed in some medium whose permittivity is not ε_0, but is instead ε? In that case, the formulas for the field become

$$E = \frac{Q}{4\pi\varepsilon r^2} \quad , \quad \frac{\lambda}{2\pi\varepsilon\, r} \quad , \quad \frac{\sigma}{2\varepsilon} \, .$$

There is an ε in the denominator of each of these expressions. When dealing with media with a permittivity other than ε_0 it is often convenient to describe the electric field by another vector, **D**, defined simply by

$$\mathbf{D} = \varepsilon\mathbf{E} \qquad 1.7.1$$

In that case the above formulas for the field become just

$$D = \frac{Q}{4\pi r^2} \quad , \quad \frac{\lambda}{2\pi\, r} \quad , \quad \frac{\sigma}{2} \, .$$

The dimensions of D are Q L^{-2}, and the SI units are C m^{-2}.

This may seem to be rather trivial, but it does turn out to be more important than it may seem at the moment.

Equation 1.7.1 would seem to imply that the electric field vectors **E** and **D** are just vectors in the same direction, differing in magnitude only by the scalar quantity ε. This is indeed the case *in vacuo* or in any isotropic medium – but it is more complicated in an anisotropic medium such as, for example, an orthorhombic crystal. This is a crystal shaped like a rectangular parallelepiped. If such a crystal is placed in an electric field, the magnitude of the permittivity depends on whether the field is applied in the *x*- , the *y*- or the *z*-direction. For a given magnitude of E, the resulting magnitude of D will be different in these three situations. And, if the field **E** is not applied parallel to one of the crystallographic axes, the resulting vector **D** will not be parallel to **E**. The permittivity in equation 1.7.1 is a *tensor* with nine components, and, when applied to **E** it changes its direction as well as its magnitude.

However, we shan't dwell on that just yet, and, unless specified otherwise, we shall always assume that we are dealing with a vacuum (in which case $\mathbf{D} = \varepsilon_0\mathbf{E}$) or an isotropic medium (in which case $\mathbf{D} = \varepsilon\mathbf{E}$). In either case the permittivity is a scalar quantity and **D** and **E** are in the same direction.

1.8 *Flux*

The product of electric field intensity and area is the *flux* Φ_E. Whereas E is an *intensive* quantity, Φ_E is an *extensive* quantity. It dimensions are $\mathrm{ML^3T^{-2}Q^{-1}}$ and its SI units are N

$m^2\ C^{-1}$, although later on, after we have met the unit called the *volt*, we shall prefer to express Φ_E in V m.

With increasing degrees of sophistication, flux may be defined mathematically as:

$$\Phi_E = EA$$

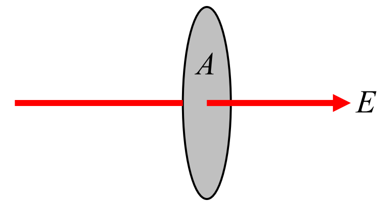


FIGURE I.4

$$\Phi_E = EA\cos\theta = \mathbf{E} \cdot \mathbf{A}$$

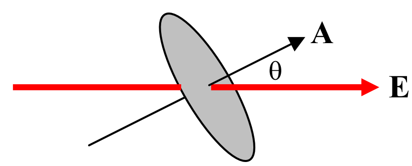


FIGURE I.5

Note that **E** is a vector, but Φ_E is a scalar.

$$\Phi_E = \iint \mathbf{E} \cdot d\mathbf{A}$$

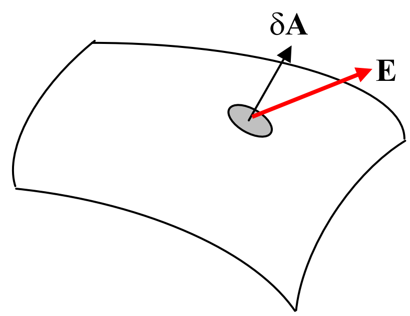


FIGURE I.6

We can also define a *D*-flux by $\Phi_D = \iint \mathbf{D} \cdot d\mathbf{A}$. The dimensions of Φ_D are just Q and the SI units are coulombs (C).

An example is in order:

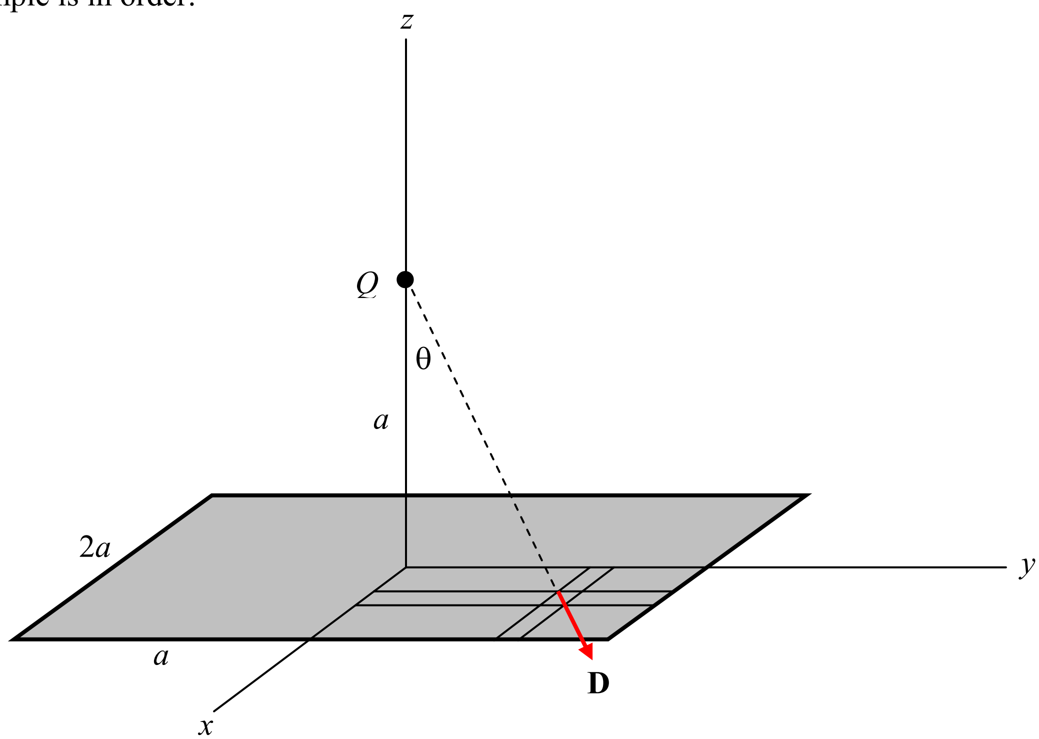


FIGURE I.7

Consider a square of side $2a$ in the xy-plane as shown. Suppose there is a positive charge Q at a height a on the z-axis. Calculate the total D-flux, Φ_D through the area.

Consider an elemental area $dxdy$ at $(x, y, 0)$. Its distance from Q is $(a^2 + x^2 + y^2)^{1/2}$, so the magnitude of the D-field there is $\frac{Q}{4\pi} \cdot \frac{1}{a^2 + x^2 + y^2}$. The scalar product of this with the area is $\frac{Q}{4\pi} \cdot \frac{1}{a^2 + x^2 + y^2} \cdot \cos\theta \, dxdy$, and $\cos\theta = \frac{a}{(a^2 + x^2 + y^2)^{1/2}}$. The surface integral of **D** over the whole area is

$$\iint D \bullet dA = \frac{Qa}{\pi} \int_0^a \int_0^a \frac{dxdy}{(a^2 + x^2 + y^2)^{3/2}} . \qquad 1.8.1$$

Now all we have to do is the nice and easy integral. Let $x = \sqrt{a^2 + y^2} \tan\psi$, and the inner integral $\int_0^a \frac{dx}{(a^2 + x^2 + y^2)^{3/2}}$ reduces, after some modest algebra, to $\frac{a}{(a^2 + y^2)\sqrt{2a^2 + y^2}}$. Thus we now have

$$\iint D \bullet dA = \frac{Qa^2}{\pi}\int_0^a \frac{dy}{(a^2 + y^2)\sqrt{2a^2 + y^2}}. \qquad 1.8.2$$

With the further substitution $a^2 + y^2 = a^2 \sec \omega$, this reduces, after more careful algebra, to

$$\iint D \bullet dA = \frac{Q}{6}. \qquad 1.8.3$$

Two additional examples of calculating surface integrals may be found in Chapter 5, section 5.6, of the Celestial Mechanics section of these notes. These deal with gravitational fields, but they are essentially the same as the electrostatic case; just substitute Q for m and $-1/(4\pi\epsilon)$ for G.

I urge readers actually to go through the pain and the algebra and the trigonometry of these three examples in order that they may appreciate all the more, in the next section, the power of Gauss's theorem.

1.9 *Gauss's Theorem*

A point charge Q is at the centre of a sphere of radius r. Calculate the D-flux through the sphere. Easy. The magnitude of D at a distance a is $Q/(4\pi r^2)$ and the surface area of the sphere is $4\pi r^2$. Therefore the flux is just Q. Notice that this is independent of r; if you double r, the area is four times as great, but D is only a quarter of what it was, so the total flux remains the same. You will probably agree that if the charge is surrounded by a shape such as shown in figure I.8, which is made up of portions of spheres of different radii, the D-flux through the surface is still just Q. And you can distort the surface as much as you like, or you may consider any surface to be made up of an infinite number of infinitesimal spherical caps, and you can put the charge anywhere you like inside the surface, or indeed you can put as many charges inside as you like – you haven't changed the total normal component of the flux, which is still just Q. This is Gauss's theorem, which is a consequence of the *inverse square* nature of Coulomb's law.

The total normal component of the D-flux through any closed surface is equal to the charge enclosed by that surface.

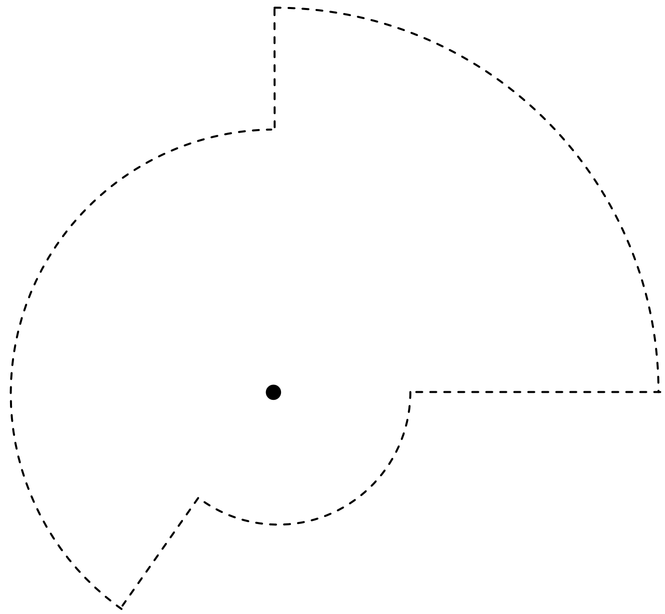


FIGURE I.8

Examples

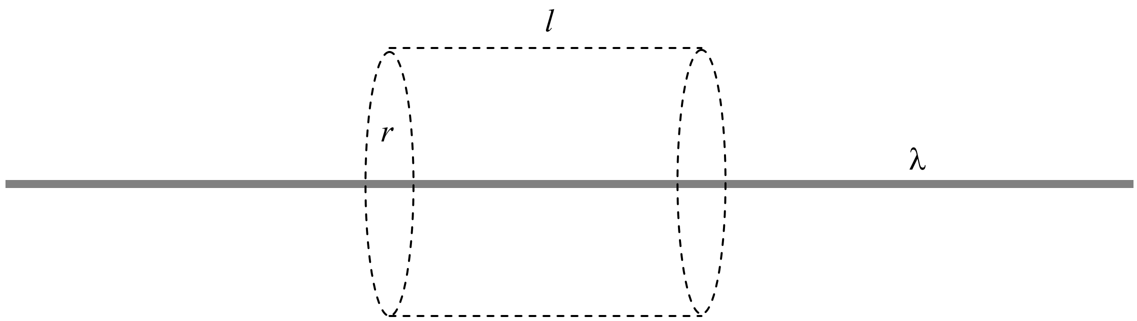


FIGURE I.9

A long rod carries a charge of λ per unit length. Construct around it a cylindrical surface of radius r and length l. The charge enclosed is $l\lambda$, and the field is directed radially outwards, passing only through the curved surface of the cylinder. The D-flux through the cylinder is $l\lambda$ and the area of the curved surface is $2\pi rl$, so $D = l\lambda/(2\pi rl)$ and hence $E = \lambda/(2\pi\varepsilon r)$.

A flat plate carries a charge of σ per unit area. Construct around it a cylindrical surface of cross-sectional area A. The charge enclosed by the cylinder is $A\sigma$, so this is the D-flux through the cylinder. It all goes through the two ends of the cylinder, which have a total area $2A$, and therefore $D = \sigma/2$ and $E = \sigma/(2\varepsilon)$.

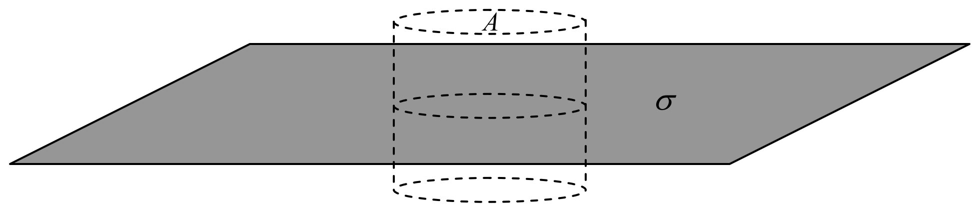


FIGURE I.10

A hollow spherical shell of radius a carries a charge Q. Construct two gaussian spherical surfaces, one of radius less than a and the other of radius $r > a$. The smaller of these two surfaces has no charge inside it; therefore the flux through it is zero, and so E is zero. The charge through the larger sphere is Q and is area is $4\pi r^2$. Therefore $D = Q/(4\pi r^2)$ and $E = Q/(4\pi\varepsilon r^2)$. (It is worth going to Chapter 5 of Celestial Mechanics, subsection 5.4.8, to go through the calculus derivation, so that you can appreciate Gauss's theorem all the more.)

A point charge Q is in the middle of a cylinder of radius a and length $2l$. Calculate the flux through the cylinder.

An infinite rod is charged with λ coulombs per unit length. It passes centrally through a spherical surface of radius a. Calculate the flux through the spherical surface.

These problems are done by calculus in section 5.6 of Celestial Mechanics, and furnish good examples of how to do surface integrals, and I recommend that you work through them. However, it is obvious from Gauss's theorem that the answers are just Q and $2a\lambda$ respectively.

A point charge Q is in the middle of a cube of side $2a$. The flux through the cube is, by Gauss's theorem, Q, and the flux through one face is $Q/6$. I hope you enjoyed doing this by calculus in section 1.8.

1.10 *A Point Charge and an Infinite Conducting Plane*

An infinite plane metal plate is in the xy-plane. A point charge $+Q$ is placed on the z-axis at a height h above the plate. Consequently, electrons will be attracted to the part of the plate immediately below the charge, so that the plate will carry a negative charge density σ which is greatest at the origin and which falls off with distance ρ from the origin. Can we determine σ(ρ)? See figure I.11

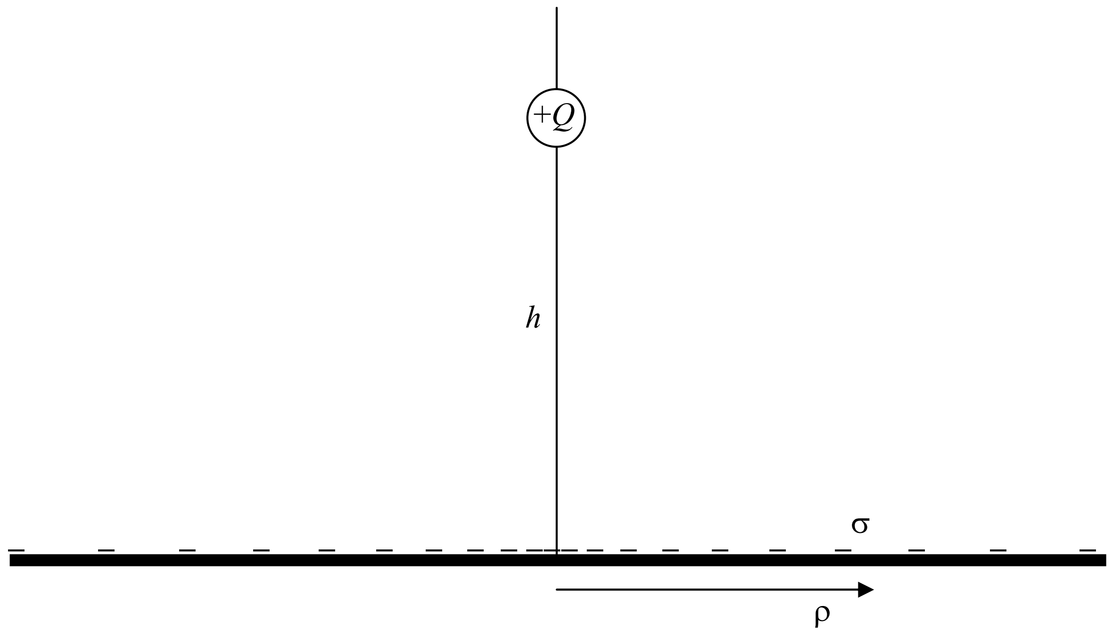


FIGURE I.11

First, note that the metal surface, being a conductor, is an *equipotential* surface, as is any metal surface. The potential is uniform anywhere on the surface. Now suppose that, instead of the metal surface, we had (in addition to the charge $+Q$ at a height h above the xy-plane), a second point charge, $-Q$, at a distance h below the xy-plane. The potential in the xy-plane would, by symmetry, be uniform everywhere. That is to say that the potential in the xy-plane is the same as it was in the case of the single point charge and the metal plate, and indeed the potential at any point above the plane is the same in both cases. For the purpose of calculating the potential, we can replace the metal plate by an *image* of the point charge. It is easy to calculate the potential at a point (z , ρ). If we suppose that the permiittivity above the plate is ε_0, the potential at (z , ρ) is

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{[\rho^2 + (h-z)^2]^{1/2}} - \frac{1}{[\rho^2 + (h+z)^2]^{1/2}}\right). \qquad 1.10.1$$

The field strength E in the xy-plane is $-\partial V/\partial z$ evaluated at $z = 0$, and this is

$$E = -\frac{2Q}{4\pi\varepsilon_0}\cdot\frac{h}{(\rho^2 + h^2)^{3/2}}\,. \qquad 1.10.2$$

The D-field is ε_0 times this, and since all the lines of force are above the metal plate, Gauss's theorem provides that the charge density is $\sigma = D$, and hence the charge density is

$$\sigma = -\frac{Q}{2\pi}\cdot\frac{h}{(\rho^2 + h^2)^{3/2}} \cdot \qquad 1.10.3$$

This can also be written $\qquad \sigma = -\frac{Q}{2\pi}\cdot\frac{h}{\xi^3}\,, \qquad 1.10.4$

where $\xi^2 = \rho^2 + h^2$, with obvious geometric interpretation.

Exercise: How much charge is there on the surface of the plate within an annulus bounded by radii ρ and $\rho + d\rho$? Integrate this from zero to infinity to show that the total charge induced on the plate is $-Q$.

1.11 *A Point Charge and a Conducting Sphere*

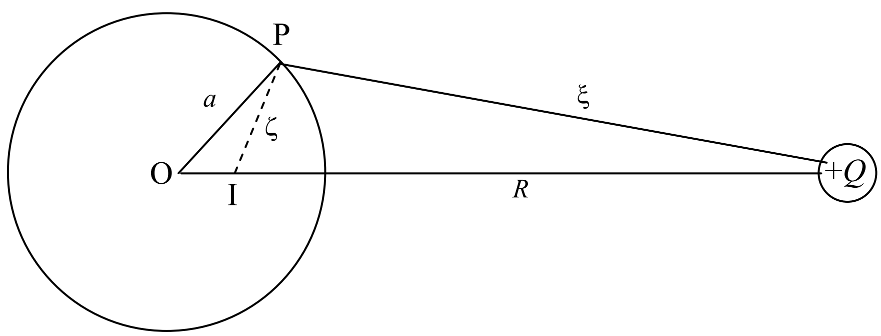


FIGURE I.12

A point charge $+Q$ is at a distance R from a metal sphere of radius a. We are going to try to calculate the surface charge density induced on the surface of the sphere, as a function of position on the surface. We shall bear in mind that the surface of the sphere is an equipotential surface, and we shall take the potential on the surface to be zero.

Let us first construct a point I such that the triangles OPI and PQO are similar, with the lengths shown in figure I.12. The length OI is a^2/R. Then $R/\xi = a/\zeta$, or

$$\frac{1}{\xi} - \frac{a/R}{\zeta} = 0. \qquad 1.11.1$$

This relation between the variables ξ and ζ is in effect the equation to the sphere expressed in these variables.

Now suppose that, instead of the metal sphere, we had (in addition to the charge $+Q$ at a distance R from O), a second point charge $-(a/R)Q$ at I. The locus of points where the potential is zero is where

$$\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\xi} - \frac{a/R}{\zeta}\right) = 0. \qquad 1.11.2$$

That is, the surface of our sphere. Thus, for purposes of calculating the potential, we can replace the metal sphere by an *image* of Q at I, this image carrying a charge of $-(a/R)Q$.

Let us take the line OQ as the z-axis of a coordinate system. Let X be some point such that OX = r and the angle XOQ = θ. The potential at P from a charge $+Q$ at Q and a charge $-(a/R)Q$ at I is (see figure I.13)

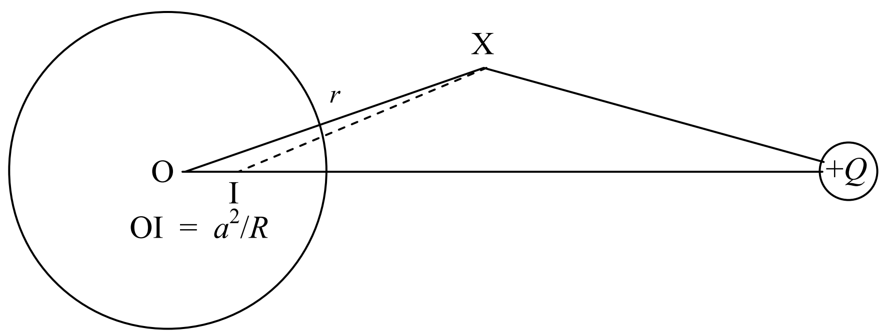


FIGURE I.13

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{(r^2+R^2-2rR\cos\theta)^{1/2}} - \frac{a/R}{(r^2+a^4/R^2-2a^2r\cos\theta/R)^{1/2}}\right). \qquad 1.11.2$$

The E field on the surface of the sphere is $-\partial V/\partial r$ evaluated at $r = a$. The D field is ε_0 times this, and the surface charge density is equal to D. After some patience and algebra, we obtain, for a point X on the surface of the sphere

$$\sigma = -\frac{Q}{4\pi}\cdot\frac{R^2 - a^2}{a}\cdot\frac{1}{(\mathrm{XQ})^3}. \qquad 1.11.3$$