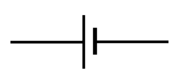
CHAPTER 4
BATTERIES, RESISTORS AND OHM'S LAW

4.1 *Introduction*

An electric *cell* consists of two different metals, or carbon and a metal, called the *poles*, immersed or dipped into a liquid or some sort of a wet, conducting paste, known as the *electrolyte*, and, because of some chemical reaction between the two poles and the electrolyte, there exists a small potential difference (typically of the order of one or two volts) between the poles. This potential difference is much smaller than the hundreds or thousands of volts that may be obtained in typical laboratory experiments in electrostatics, and the electric field between the poles is also correspondingly small.

Definition. The potential difference across the poles of a cell when no current is being taken from it is called the *electromotive force* (EMF) of the cell.

The circuit symbol for a cell is drawn thus:



The longer, thin line represents the positive pole and the shorter, thick line represents the negative pole.

Several cells connected together form a *battery* of cells. Thus in principle a single cell should strictly be called just that – a *cell* – and the word *battery* should be restricted to a battery of several cells. However, in practice, most people use the word battery to mean either literally a battery of several cells, or a single cell.

I shall not discuss in this chapter the detailed chemistry of why there exists such a potential difference, nor shall I discuss in detail the chemical processes that take place inside the several different varieties of cell. I shall just mention that in the cheaper types of flashlight battery (cell), the negative pole, made of zinc, is the outer casing of the cell, while the positive pole is a central carbon rod. The rather dirty mess that is the electrolyte is a mixture that is probably known only to the manufacturer, though it probably includes manganese oxide and ammonium chloride and perhaps such goo as flour or glue and goodness knows what else. Other types have a positive pole of nickelic hydroxide and a negative pole of cadmium metal in a potassium hydroxide electrolyte. A 12-volt car battery is typically a battery of 6 cells in series, in which the positive poles are lead oxide PbO_2, the negative poles are metallic lead and the electrolyte is sulphuric acid. In some batteries, after they are exhausted, the poles are irreversibly damaged and the battery has to be discarded. In others, such as the nickel-cadmium or lead-acid cells, the chemical reaction is reversible, and so the cells can be recharged. I have heard the word "accumulator" used for a rechargeable battery, particularly the lead-acid car battery, but I don't know how general that usage is.

Obviously the purpose of a battery is to extract a current from it. An *electrolytic cell* is quite the opposite. In an electrolytic cell, an electric current is forced into it from outside. This may be done in a laboratory, for example, to study the flow of electricity through an electrolyte, or in industrial processes such as electroplating. In an electrolytic cell, the current is forced into the cell by two *electrodes*, one of which (the *anode*) is maintained at a higher potential than the other (the *cathode*). The electrolyte contains positive ions (*cations*) and negative ions (*anions*), which can flow through the electrolyte. Naturally, the positive ions (cations) flow towards the negative electrode (the cathode) and the negative ions (the anions) flow towards the positive electrode (the anode).

The direction of flow of electricity in an electrolytic cell is the opposite from the flow when a battery is being used to power an external circuit, and the roles of the two poles or electrodes are reversed. Thus some writers will refer to the *positive* pole of a *battery* as its "cathode". It is not surprising therefore, that many a student (and, one might even guess, many a professor and textbook writer) has become confused over the words cathode and anode. The situation is not eased by referring to negatively charged electrons in a gaseous discharge tube as "cathode rays".

My recommendation would be: When referring to an electrolytic cell, use the word "electrodes"; when referring to a battery, use the word "poles". Avoid the use of the prefixes "cat" and "an" altogether. Thus, refer to the positive and negative electrodes of an electrolytic cell, the positive and negative poles of a battery, and the positive and negative ions of an electrolyte. In that way your meaning will always be clear and unambiguous to yourself and to your audience or your readers.

4.2 *Resistance and Ohm's Law*

When a potential difference is maintained across the electrodes in an electrolytic cell, a current flows through the electrolyte. This current is carried by positive ions moving from the positive electrode towards the negative electrode and also, simultaneously, by negative ions moving from the negative electrode towards the positive electrode. The conventional direction of the flow of electricity is the direction in which positive charges are moving. That is to say, electricity flows from the positive electrode towards the negative electrode. The positive ions, then, are moving in the same direction as the conventional direction of flow of electricity, and the negative ions are moving in the opposite direction.

When current flows in a *metal*, the current is carried exclusively by means of negatively charged electrons, and therefore the current is carried exclusively by means of particles that are moving in the opposite direction to the conventional flow of electricity. Thus "electricity" flows from a point of high potential to a point of lower potential; electrons move from a point of low potential to a point of higher potential.

When a potential difference V is applied across a resistor, the ratio of the potential difference across the resistor to the current I that flows through it is called the *resistance*, R, of the resistor. Thus

$$V = IR. \qquad 4.2.1$$

This equation, which defines resistance, appears at first glance to say that *the current through a resistor is proportional to the potential difference across it*, and this is *Ohm's Law.* Equation 4.2.1, however, implies a simple proportionality between V and I only if R is constant and independent of I or of V. In practice, when a current flows through a resistor, the resistor becomes hot, and its resistance increases – and then V and I are no longer linearly proportional to one another. Thus one would have to state Ohm's Law in the form that *the current through a resistor is proportional to the potential difference across it, provided that the temperature is held constant*. Even so, there are some substances (and various electronic devices) in which the resistance is not independent of the applied potential difference even at constant temperature. Thus it is better to regard equation 4.2.1 as a definition of resistance rather than as a fundamental law, while also accepting that it is a good description of the behaviour of most real substances under a wide variety of conditions as long as the temperature is held constant.

Definitions. If a current of one amp flows through a resistor when there is a potential difference of one volt across it, the resistance is one *ohm* (Ω). (Clear though this definition may appear, however, recall from chapter 1 that we have not yet defined exactly what we mean by an amp, nor a volt, so suddenly the meaning of "ohm" becomes a good deal less clear! I do promise a definition of "amp" in a later chapter – but in the meantime I crave your patience.)

The *dimensions* of resistance are $\dfrac{\mathrm{ML^2T^{-2}Q^{-1}}}{\mathrm{T^{-1}Q}} = \mathrm{ML^2T^{-1}Q^{-2}}$.

The reciprocal of resistance is *conductance*, G. Thus $I = GV$. It is common informal practice to express conductance in "mhos", a "mho" being an ohm^{-1}. The official SI unit of conductance, however, is the siemens (S), which is the same thing as a "mho", namely one A V^{-1}.

The *resistance* of a *resistor* is proportional to its length l and inversely proportional to its cross-sectional area A:

$$R = \frac{\rho l}{A}. \qquad 4.2.2$$

The constant of proportionality ρ is called the *resistivity* of the material of which the resistor is made. Its dimensions are $\mathrm{ML^3T^{-1}Q^{-2}}$, and its SI unit is ohm metre, or Ω m.

The reciprocal of resistivity is the *conductivity*, σ. Its dimensions are $M^{-1}L^{-3}TQ^2$, and its SI unit is siemens per metre, S m^{-1}.

For those who enjoy collecting obscure units, there is an amusing unit I once came across, namely the unit of surface resistivity. One is concerned with the resistance of a thin sheet of conducting material, such as, for example, a thin metallic film deposited on glass. The resistance of some rectangular area of this is proportional to the length l of the rectangle and inversely proportional to its width w:

$$R = \frac{\rho l}{w}.$$

The resistance, then, depends on the ratio l/w – i.e. on the shape of the rectangle, rather than on its size. Thus the resistance of a 2 mm × 3 mm rectangle is the same as that of a 2 m × 3 m rectangle, but quite different from that of a 3 mm × 2 mm rectangle. The surface resistivity is defined as the resistance of a rectangle of unit length and unit width (i.e. a square) – and it doesn't matter what the size of the square. Thus the units of surface resistivity are ohms per square. (End of sentence!)

As far as their resistivities are concerned, it is found that substances may be categorized as *metals*, *nonconductors* (insulators), and *semiconductors*. Metals have rather low resistivities, of the order of 10^{-8} Ω m. For example:

Silver:	1.6×10^{-8} Ω m
Copper:	1.7×10^{-8}
Aluminium:	2.8×10^{-8}
Tungsten:	5.5×10^{-8}
Iron:	10×10^{-8}

Nonconductors have resistivities typically of order 10^{14} to 10^{16} Ω m or more. That is, for most practical purposes and conditions they don't conduct any easily measurable electricity at all.

Semiconductors have intermediate resistivities, such as

Carbon:	1500×10^{-8} Ω m
Germanium:	4.5×10^{-1}
Silicon	$6.4 \times 10^{+2}$

There is another way, besides equation 4.2.1, that is commonly used to express Ohm's law. Refer to figure IV.1.

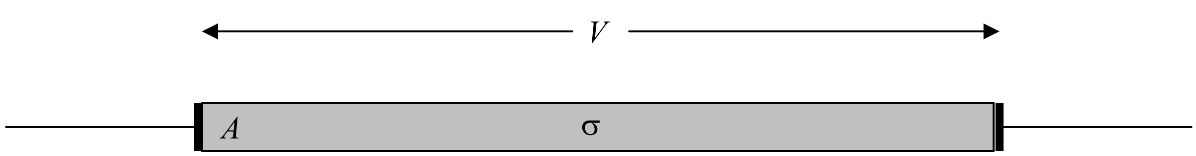


FIGURE IV.1

We have a metal rod of length l, cross-sectional area A, electrical conductivity σ, and so its resistance is $l/(\sigma A)$. We clamp it between two points which have a potential difference of V between them, and consequently the magnitude of the electric field in the metal is $E = V/l$. Equation 4.2.1 (Ohm's law) therefore becomes $El = Il/(\sigma A)$. Now introduce $J = I/l$ as the *current density* (amps per square metre). Them Ohm's law becomes $J = \sigma E$. This is usually written in vector form, since current and field are both vectors, so that Ohm's law is written

$$\mathbf{J} = \sigma\mathbf{E}. \qquad 4.2.3$$

4.3 *Resistance and Temperature*

It is found that the resistivities of metals generally increase with increasing temperature, while the resistivities of semiconductors generally decrease with increasing temperature.

It may be worth thinking a little about how electrons in a metal or semiconductor conduct electricity. In a solid metal, most of the electrons in an atom are used to form covalent bonds between adjacent atoms and hence to hold the solid together. But about one electron per atom is not tied up in this way, and these "conduction electrons" are more or less free to move around inside the metal much like the molecules in a gas. We can estimate roughly the speed at which the electrons are moving. Thus we recall the formula $\sqrt{3kT/m}$ for the root-mean-square speed of molecules in a gas, and maybe we can apply that to electrons in a metal just for a rough order of magnitude for their speed. Boltzmann's constant k is about 1.38×10^{-23} J K^{-1} and the mass of the electron, m, is about 9.11×10^{-31} kg. If we assume that the temperature is about 27°C or 300 K, the root mean square electron speed would be about 1.2×10^{5} m s^{-1}.

Now consider a current of 1 A flowing in a copper wire of diameter 1 mm – i.e. cross-sectional area 7.85×10^{-7} m^2. The density of copper is 8.9 g cm^{-3}, and its "atomic weight" (molar mass) is 63.5 g per mole, which means that there are 6.02×10^{23} (Avogadro's number) of atoms in 63.5 grams, or 8.44×10^{22} atoms per cm^3 or 8.44×10^{28} atoms per m^3. If we assume that there is one conduction electron per atom, then there are 8.44×10^{28} conduction electrons per m^3, or, in our wire of diameter 1 mm, 6.63×10^{22} conduction electrons per metre.

The speed at which the electrons are carrying the current of one amp is the current divided by the charge per unit length, and with the charge on a single electron being 1.60×10^{-19} C, we find that the speed at which the electrons are carrying the current is about 9.4×10^{-5} m s^{-1}.

Thus we have this picture of electrons moving in random directions at a speed of about 1.2×10^{5} m s^{-1} (the thermal motion) and, superimposed on that, a very slow drift speed of only 9.4×10^{-5} m s^{-1} for the electron current. If you were able to see the electrons, you would see them dashing hither and thither at very high speeds, but you wouldn't even notice the very slow drift in the direction of the current.

When you connect a long wire to a battery, however, the current (the slow electron drift) starts almost instantaneously along the entire length of the wire. If the electrons were in a complete vacuum, rather than in the interior of a metal, they would accelerate as long as they were in an electron field. The electrons inside the metal also accelerate, but they are repeatedly stopped in their tracks by collisions with the metal atoms – and then they start up again. If the temperature is increased, the vibrations of the atoms within the metal lattice increase, and this presumably somehow increases the resistance to the electron flow, or decreases the mean time or the mean path-length between collisions.

In a *semiconductor*, most of the electrons are required for valence bonding between the atoms – but there are a few (much fewer than one per atom) free, conduction electrons. As the temperature is increased, more electrons are shaken free from their valence duties, and they then take on the task of conducting electricity. Thus the conductivity of a semiconductor increases with increasing temperature.

The temperature coefficient of resistance, α, of a metal (or other substance) is the fractional increase in its resistivity per unit rise in temperature:

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}. \tag{4.3.1}$$

In SI units it would be expressed in K^{-1}. However, in many practical applications the temperature coefficient is defined in relation to the change in resistance compared with the resistivity at a temperature of 20°C, and is given by the equation

$$\rho = \rho_{20}[1 + \alpha(t - 20)], \tag{4.3.2}$$

where t is the temperature in degrees Celsius.

Examples:

Silver:	3.8×10^{-3} C° $^{-1}$
Copper:	3.9×10^{-3}
Aluminium:	3.9×10^{-3}
Tungsten:	4.5×10^{-3}
Iron:	5.0×10^{-3}
Carbon:	-0.5×10^{-3} C° $^{-1}$
Germanium:	-48×10^{-3}
Silicon	-75×10^{-3}

Some metallic alloys with commercial names such as nichrome, manganin, constantan, eureka, etc., have fairly large resistivities and very low temperature coefficients.

As a matter of style, note that the *kelvin* is a unit of temperature, much a the *metre* is a unit of length. Thus, when discussing temperatures, there is no need to use the "degree" symbol with the kelvin. When you are talking about some other temperature scale, such as Celsius, one needs to say "20 degrees on the Celsius scale" – thus 20ºC. But when one is talking about a temperature *interval* of so many Celsius degrees, this is written Cº. I have adhered to this convention above.

The resistivity of platinum as a function of temperature is used as the basis of the *platinum resistance thermometer*, useful under conditions and temperatures where other types of thermometers may not be useful, and it is also used for defining a practical temperature scale at high temperatures. A *bolometer* is an instrument used for detecting and measuring infrared radiation. The radiation is focussed on a blackened platinum disc, which consequently rises in temperature. The temperature rise is measured by measuring the increase in resistance. A *thermistor* is a semiconducting device whose resistance is very sensitive to temperature, and it can be used for measuring or controlling temperature.

4.4 *Resistors in Series*

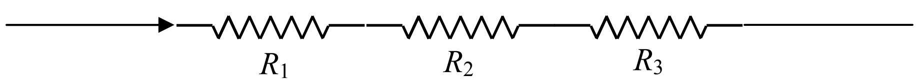


FIGURE IV.2

$$R = R_1 + R_2 + R_3 . \qquad 4.4.1$$

The current is the same in each. The potential difference is greatest across the largest resistance.

4.5 *Conductors in Parallel*

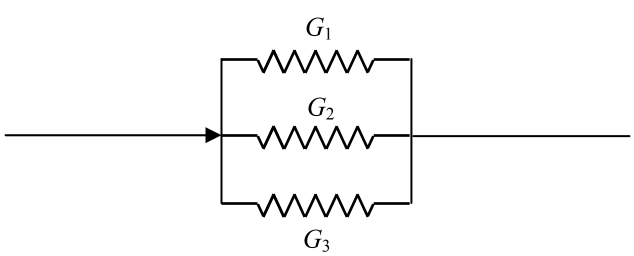


FIGURE IV.3

$$G = G_1 + G_2 + G_3 . \qquad 4.5.1$$

That is to say $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}. \qquad 4.5.2$$

The potential difference is the same across each. The current is greatest through the largest conductance – i.e. through the smallest resistance.

4.6 *Dissipation of Energy*

When current flows through a resistor, electricity is falling through a potential difference. When a coulomb drops through a volt, it loses potential energy 1 joule. This energy is dissipated as heat. When a current of I coulombs per second falls through a potential difference of V volts, the rate of dissipation of energy is IV, which can also be written (by making use of Ohm's law) I^2R or V^2/R.

If two resistors are connected in series, the current is the same in each, and we see from the formula I^2R that more heat is generated in the larger resistance.

If two resistors are connected in parallel, the potential difference is the same across each, and we see from the formula V^2/R that more heat is generated in the small resistance.

4.7 *Electromotive Force and Internal Resistance*

The reader is reminded of the following definition from section 4.1:

Definition. The potential difference across the poles of a cell when no current is being taken from it is called the *electromotive force* (EMF) of the cell.

I shall use the symbol $\mathcal{E}$ for EMF.

Question. A 4 Ω resistance is connected across a cell of EMF 2 V. What current flows? The immediate answer is 0.5 A – but this is likely to be wrong. The reason is that a cell has a resistance of its own – its *internal resistance.* The internal resistance of a lead-acid cell is typically quite small, but most dry cells have an appreciable internal resistance. If the external resistance is R and the internal resistance is r, the total resistance of the circuit is $R + r$, so that the current that flows is $\mathcal{E}/(R + r)$.

Whenever a current is taken from a cell (or battery) the potential difference across its poles *drops* to a value less than its EMF. We can think of a cell as an EMF in series with an internal resistance:

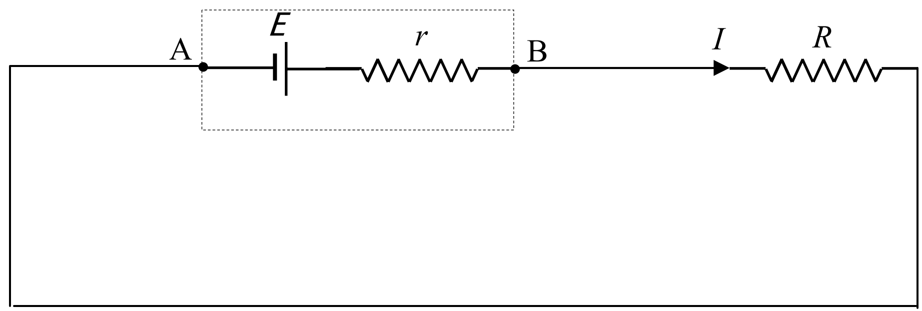


FIGURE IV.4

If we take the point A as having zero potential, we see that the potential of the point B will be $E - Ir$, and this, then, is the potential difference across the poles of the cell when a current I is being taken from it.

Exercise. Show that this can also be written as $\dfrac{ER}{R + r}$.

4.8 *Power Delivered to an External Resistance*

Question: How much heat will be generated in the external resistance R if $R = 0$?
Answer: None!

Question: How much heat will be generated in the external resistance R if $R = \infty$?
Answer: None!

Question: How much heat will be generated in the external resistance R if R is something?
Answer: Something!

This suggests that there will be some value of the external resistance for which the power delivered, and heat generated, will be a maximum, and this is indeed the case.

The rate at which power is delivered, and dissipated as heat, is

$$P = I^2 R = \frac{E^2 R}{(R + r)^2} \cdot \qquad 4.8.1$$

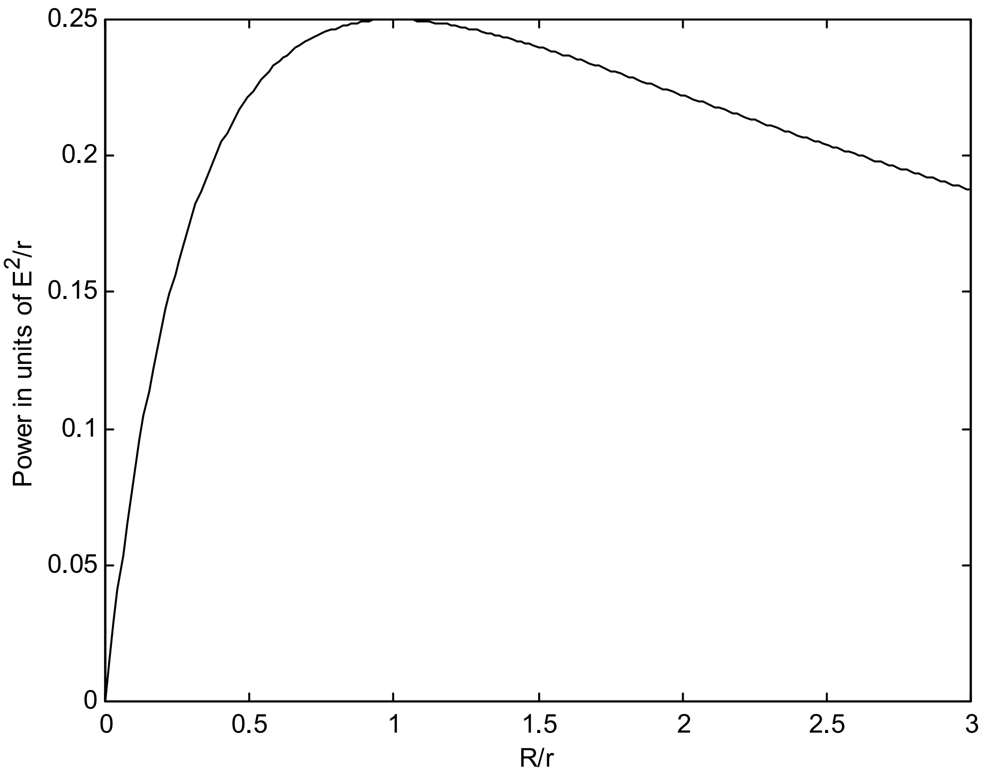
In figure IV.5 I have plotted the power (in units of E^2/R) versus R/r. Differentiation of the above expression (do it!) will show that the power delivered reaches a maximum of

$E^2/(4R)$ when $R = r$; that is, when the external resistance is "matched" to the internal resistance of the cell. This is but one example of many in physics and engineering in which maximum power is delivered to a load when the load is matched to the internal load of the power source.

Exercise. A 6V battery with an internal resistance of 0.5 Ω is connected to an external resistance. Heat is generated in the external resistance at a rate of 12 W. What is the value of the external resistance?

Answer. 1.87 Ω or 0.134 Ω.

FIGURE IV.5



4.9 *Potential Divider*

The circuit illustrated in figure IV.6 is a *potential divider*.

It may be used to supply a variable voltage to an external circuit. It is then called a *rheostat*.

Or it may be used to compare potential differences, in which case it is called a *potentiometer*. (In practice many people refer to such a device as a “pot”, regardless of the use to which it is put.)

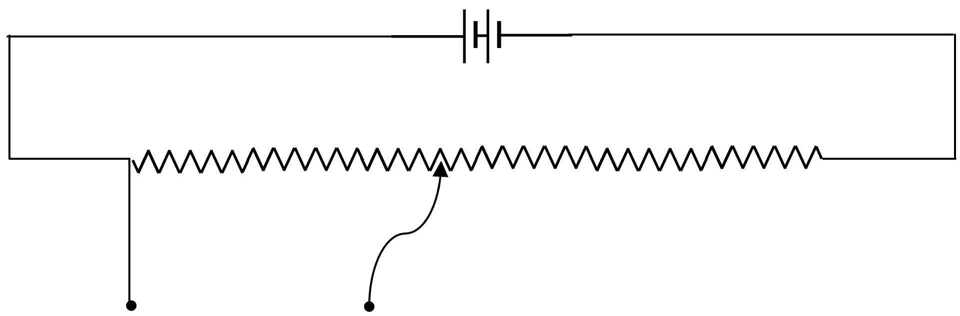


FIGURE IV.6

For example, in figure IV.7, a balance point (no current in the ammeter, A) is found when the potential drop down the length x of the resistance wire is equal to the EMF of the small cell. (Note that, since no current is being taken from the small cell, the potential difference across its poles is indeed the EMF.) One could compare the EMFs of two cells in this manner, one of which might be a “standard cell” whose EMF is known.

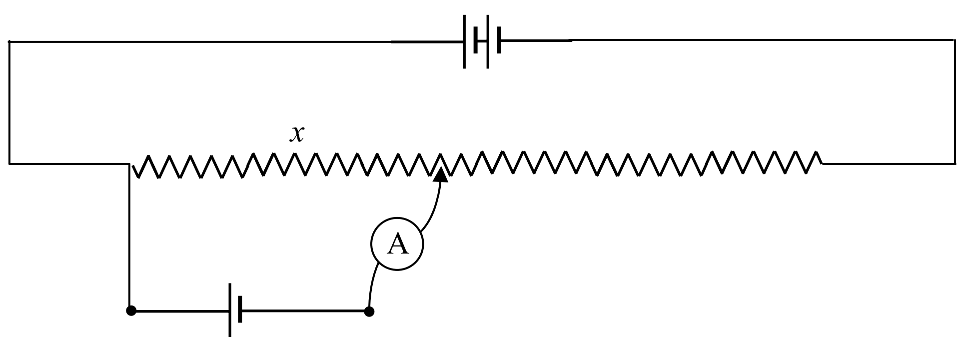


FIGURE IV.7

In figure IV.8, a current is flowing through a resistor (which is assumed to be in part of some external circuit, not drawn), and, assuming that the potential gradient down the potentiometer has been calibrated with a standard cell, the potentiometer is being used to measure the potential difference across the resistor. That is, the potentiometer is being used as a voltmeter.

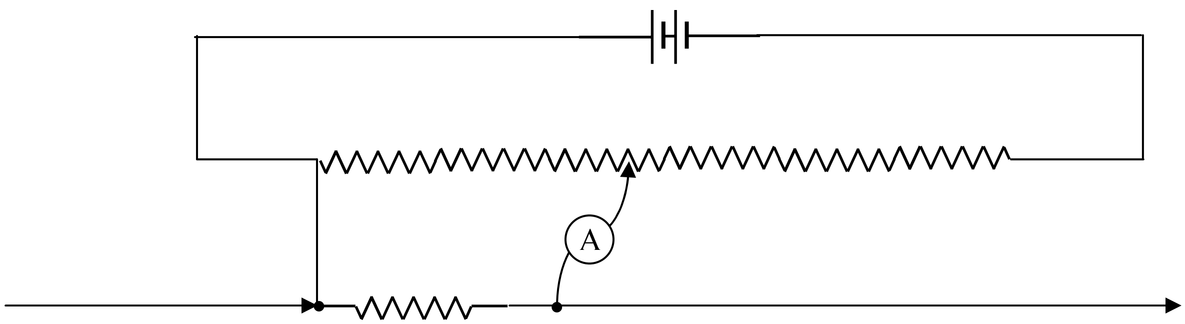


FIGURE IV.8

4.10 *Ammeters and Voltmeters*

For the purpose of this section it doesn't matter how an ammeter actually works. Suffice it to say that a current flows through the ammeter and a needle moves over a scale to indicate the current, or else the current is indicated as numbers in a digital display. In order to measure the current through some element of a circuit, the ammeter is placed, of course, in *series* with the element. Generally an ammeter has rather a low resistance.

An inexpensive *voltmeter* is really just an ammeter having rather a high resistance. If you want to measure the potential difference across some circuit element, you place the voltmeter, of course, *across* that element (i.e. in *parallel* with it). A small portion of the current through the element is diverted through the meter; the meter measures this current, and, from the known resistance of the meter, the potential difference can be calculated – though in practice nobody does any calculation – the scale is marked in volts. Placing a meter *across* a circuit element in fact slightly reduces the potential difference across the element – that is, it reduces the very thing you want to measure. But, because a voltmeter typically has a high resistance, this effect is small. There are, of course, modern (and more expensive) voltmeters of a quite different design, which take no current at all, and genuinely measure potential difference, but we are concerned in this section with the commonly-encountered ammeter-turned-voltmeter. It may be noticed that the potentiometer described in the previous section takes no current from the circuit element of interest, and is therefore a true voltmeter.

There are meters known as "multimeters" or "avometers" (for amps, volts and ohms), which can be used as ammeters or as voltmeters, and it is with these that this section is concerned.

A typical inexpensive ammeter gives a full scale deflection (FSD) when a current of 15 mA = 0.015 A flows through it. It can be adapted to measure higher currents by connecting a small resistance (known as a "shunt") *across* it.

Let's suppose, for example, that we have a meter that which shows a FSD when a current of 0.015 A flows through it, and that the resistance of the meter is 10 Ω. We would like to use the meter to measure currents as high as 0.15 A. What value of shunt resistance shall we put across the meter? Well, when the total current is 0.15 A, we want 0.015 A to flow through the meter (which then shows FSD) and the remainder, 0.135 A, is to flow through the shunt. With a current of 0.015 A flowing through the 10 Ω meter, the potential difference across it is 0.15 V. This is also the potential difference across the shunt, and, since the current through the shunt is 0.135 A, the resistance of the shunt must be 1.11 Ω.

We can also use the meter as a voltmeter. Suppose, for example, that we want to measure voltages (horrible word!) of up to 1.5 V. We place a large resistance R in *series* with the meter, and then place the meter-plus-series-resistance across the potential difference to measured. The total resistance of meter-plus-series-resistance is $(10 + R)$, and it will show a FSD when the current through it is 0.015 A. We want this to happen when the potential difference across it is 1.5 volts. This $1.5 = 0.015 \times (10 + R)$, and so $R = 90$ Ω.

4.11 *Wheatstone Bridge*

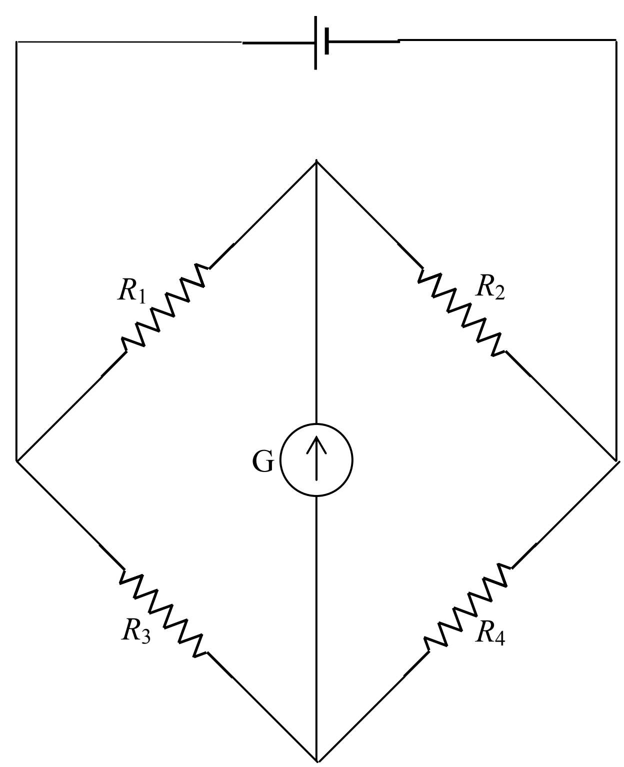


FIGURE IV.9

The Wheatstone bridge can be used to *compare* the value of two resistances – or, if the unknown resistance is compared with a resistance whose value is known, it can be used to *measure* an unknown resistance. R_1 and R_2 can be varied. R_3 is a standard resistance whose value is known. R_4 is the unknown resistance whose value is to be determined. G is a *galvanometer*. This is just a sensitive ammeter, in which the zero-current position has the needle in the middle of the scale; the needle may move one way or the other, depending on which way the current is flowing. The function of the galvanometer is not so much to *measure* current, but merely to *detect* whether or not a current is flowing in one direction of another. In use, the resistances R_1 and R_2 are varied until no current flows in the galvanometer. The bridge is then said to be "balanced" and $R_1 / R_2 = R_3 / R_4$, and hence the unknown resistance is given by $R_4 = R_1 R_3 / R_2$.

4.12 *Delta-Star Transform*

Consider the two circuits (each enclosed in a black box) of figure IV.10.

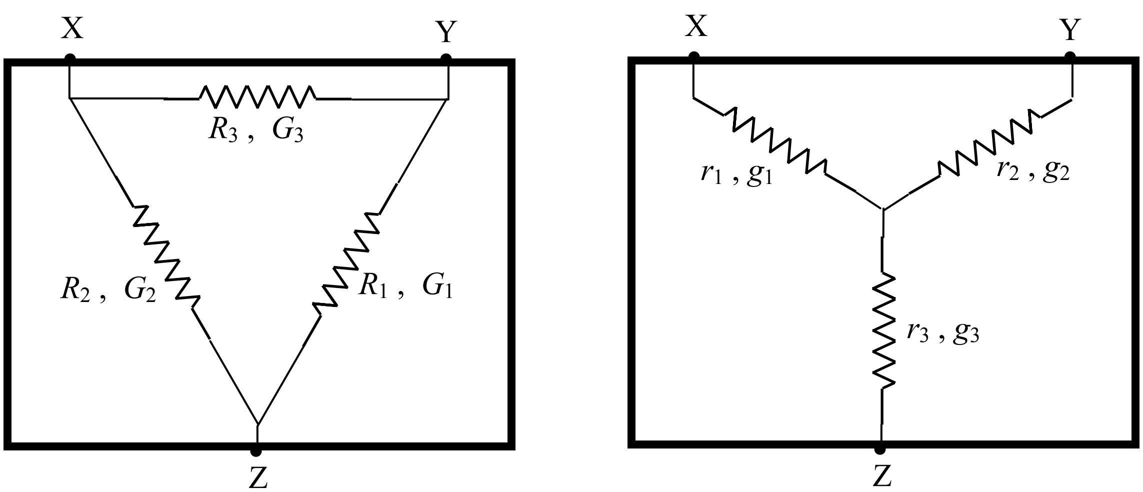


FIGURE IV.10

The configuration in the left hand box is called a "delta" (Δ) and the configuration in the right hand box is called a "star" or a "Y". I have marked against each resistor its resistance and its conductance, the conductance, of course, merely being the reciprocal of the resistance. I am going to suppose that the resistance between the terminals X and Y is the same for each box. In that case:

$$r_1 + r_2 = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} . \qquad 4.12.1$$

We can get similar equations for the terminal pairs Y,Z and Z,X. Solving the three equations for r_1 , r_2 and r_3, we obtain

$$r_1 = \frac{R_2 R_3}{R_1 + R_2 + R_3}, \qquad 4.12.2$$

$$r_2 = \frac{R_3 R_1}{R_1 + R_2 + R_3} \qquad 4.12.3$$

and $$r_3 = \frac{R_1 R_2}{R_1 + R_2 + R_3}. \qquad 4.12.4$$

In terms of the conductances, these are

$$g_1 = \frac{G_2 G_3 + G_3 G_1 + G_1 G_2}{G_1}, \qquad 4.12.5$$

$$g_2 = \frac{G_2 G_3 + G_3 G_1 + G_1 G_2}{G_2} \qquad 4.12.6$$

and $$g_3 = \frac{G_2 G_3 + G_3 G_1 + G_1 G_2}{G_3}. \qquad 4.12.7$$

The converses of these equations are:

$$R_1 = \frac{r_2 r_3 + r_3 r_1 + r_1 r_2}{r_1}, \qquad 4.12.8$$

$$R_2 = \frac{r_2 r_3 + r_3 r_1 + r_1 r_2}{r_2}, \qquad 4.12.9$$

$$R_3 = \frac{r_2 r_3 + r_3 r_1 + r_1 r_2}{r_3}, \qquad 4.12.10$$

$$G_1 = \frac{g_2 g_3}{g_1 + g_2 + g_3}, \qquad 4.12.11$$

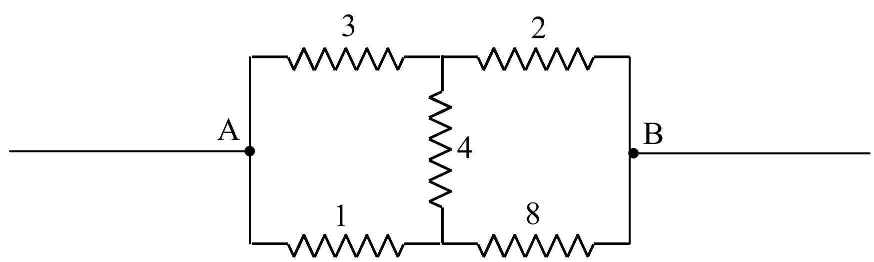
$$G_2 = \frac{g_3 g_1}{g_1 + g_2 + g_3} \qquad 4.12.12$$

and $$G_3 = \frac{g_1 g_2}{g_1 + g_2 + g_3}. \qquad 4.12.13$$

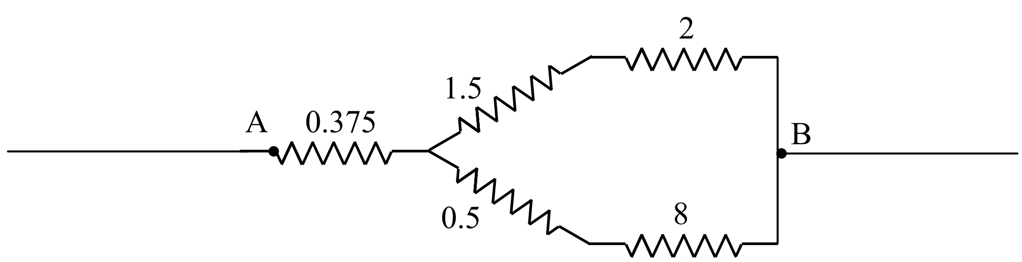
That means that, if the resistances and conductances in one box are related to the resistances and conductances in the other by these equations, then you would not be able to tell, if you had an ammeter, and a voltmeter and an ohmmeter, which circuit was in which box. The two boxes are indistinguishable from their electrical behaviour.

These equations are not easy to commit to memory unless you are using them every day, and they are sufficiently awkward that mistakes are likely when evaluating them numerically. Therefore, to make the formulas useful, you should programme your calculator or computer so that they will instantly convert between delta and star without your ever having to think about it. The next example shows the formulas in use. It will be heavy work unless you have programmed your computer in advance – but if you *have* done so, you will see how very useful the transformations are.

Example. Calculate the resistance between the points A and B in the figure below. The individual resistances are given in ohms.



At first, one doesn't know how to start. But notice that the 1, 3 and 4 ohm resistors are connected in delta and the circuit is therefore equivalent to



After that, it is easy, and you will soon find that the resistance between A and B is 2.85 Ω.

4.13 *Kirchhoff's Rules*

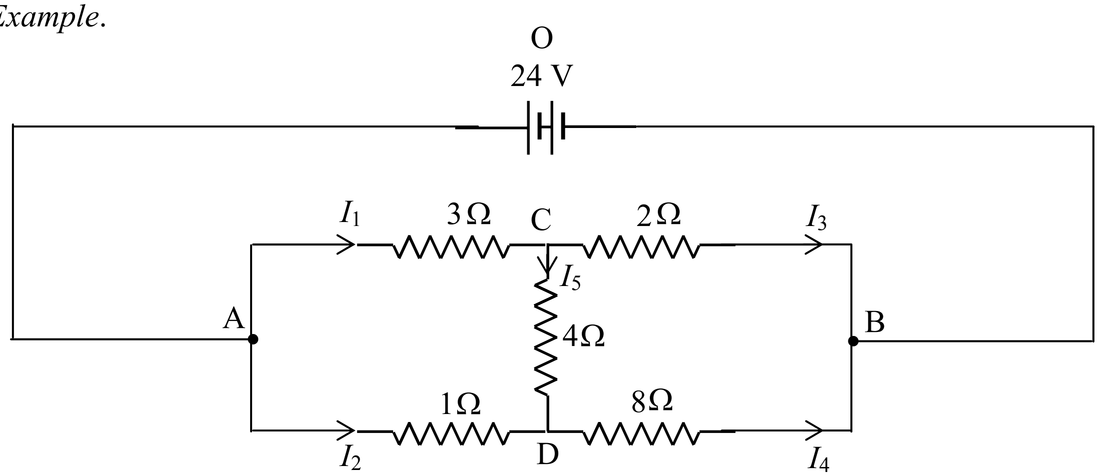
There are two h's in his name, and there is no *tch* sound in the middle. The pronunciation is approximately keerr–hhofe.

The rules themselves are simple and are self-evident. What has to be learned, however, is the art of using them.

K1: The net current going into any *point* in a circuit is zero; expressed otherwise, the sum of all the currents entering any point in a circuit is equal to the sum of all the currents leaving the point.

K2: The sum of all the EMFs and *IR* products in a *closed circuit* is zero. Expressed otherwise, as you move around a closed circuit, the potential will sometimes rise and sometimes fall as you encounter a battery or a resistance; but, when you come round again to the point where you started, there is no change in potential.

Example.



In the above circuit, the 24 V battery is assumed to have negligible internal resistance. Calculate the current in each of the resistors.

The art of applying Kirchhoff's rules is as follows.

1. Draw a large circuit diagram *in pencil*.

2. Count the number of independent resistors. (Two is series with nothing in between don't count as independent.) This tells you how many independent equations you can obtain, and how many unknowns you can solve for. In this case, there are five independent resistors; you can get five independent equations and you can solve for five unknowns.

3. Mark in the unknown currents. If you don't know the directions of some of them, don't spend time trying to think it out. Just make a wild guess. If you are wrong, you will merely get a negative answer for it. Those who have some physical insight might already guess (correctly) that I have marked I_5 in the wrong direction, but that doesn't matter.

4. Choose any closed circuit and apply K2. Go over that closed surface *in ink*. Repeat for several closed circuits until the entire diagram is inked over. When this happens, you cannot get any further independent equations using K2. If you try to do so, you will merely end up with another equation that is a linear combination of the ones you already have,

5. Make up the required number of equations with K1.

Let us apply these to the present problem. There are five resistors; we need five equations. Apply K2 to OACBO. Start at the negative pole of the battery and move counterclockwise around the circuit. When we move up to the positive pole, the potential has gone up by 24 V. When we move down a resistor in the direction of the current, the potential goes down. For the circuit OACBO, K2 results in

$$24 - 3I_1 - 2I_3 = 0.$$

Now do the same this with circuit OADBO:

$$24 - I_2 - 8I_4 = 0,$$

and with circuit ACDA:

$$3I_1 + 4I_5 - I_2 = 0.$$

If you have conscientiously inked over each circuit as you have done this, you will now find that the entire diagram is inked over. You cannot gain any further independent equations from K2. We need two more equations. Apply K1 to point C:

$$I_1 = I_3 + I_5,$$

and to point D: $$I_4 = I_2 + I_5.$$

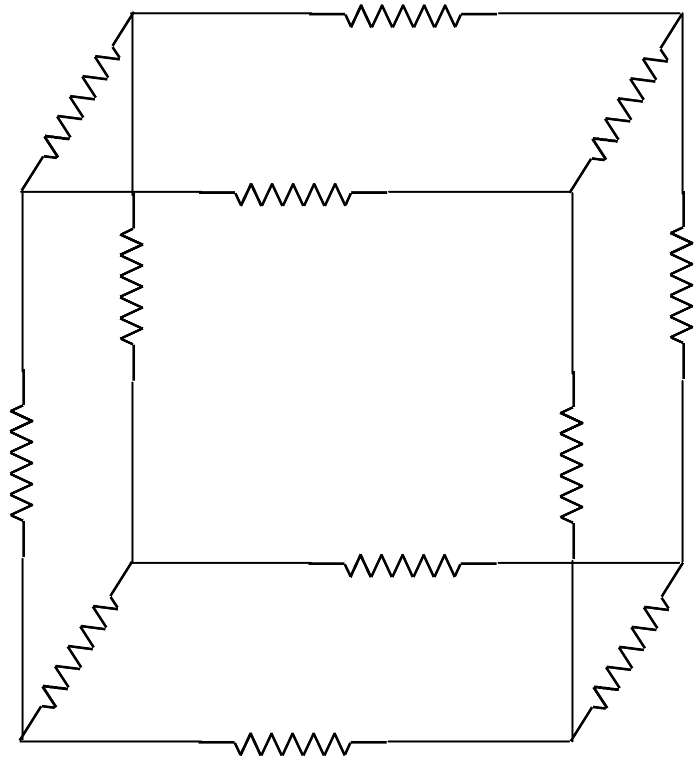
You now have five independent linear equations in five unknowns and you can solve them. (Methods for solving simultaneous linear equations are given in Chapter 1, Section 1.7 of Celestial Mechanics.) The solutions are:

$$I_1 = +4.029\text{A},\ I_2 = +4.380\text{A},\ I_3 = +5.956\text{A},\ I_4 = +2.453\text{A},\ I_5 = -1.927\text{A}.$$

4.14 *Tortures for the Brain*

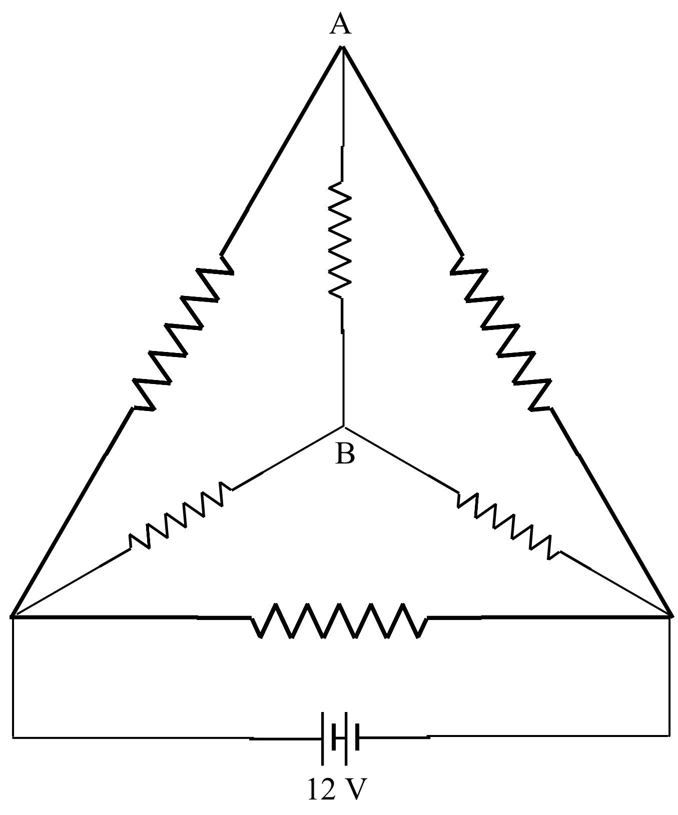
I don't know if any of the examples in this section have any practical applications, but they are excellent ways for torturing students, or for whiling away rainy Sunday afternoons.

4.14.1



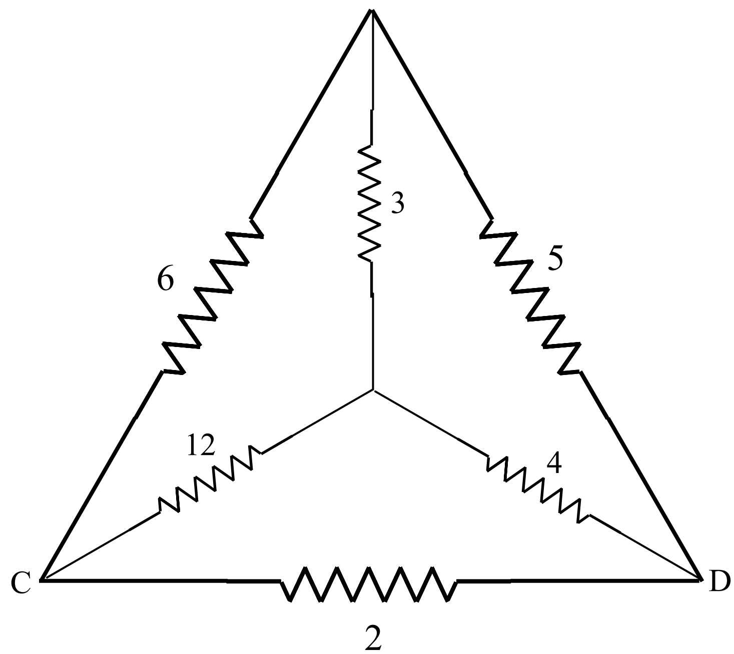
The drawing shows 12 resistances, each of value $r\ \Omega$, arranged along the edges of a cube. What is the resistance across opposite corners of the cube?

4.14.2



The drawing shows six resistors, each of resistance 1 Ω, arranged along the edges of a tetrahedron. A 12 V battery is connected across one of the resistors. Calculate the current between points A and B.

4.14.3

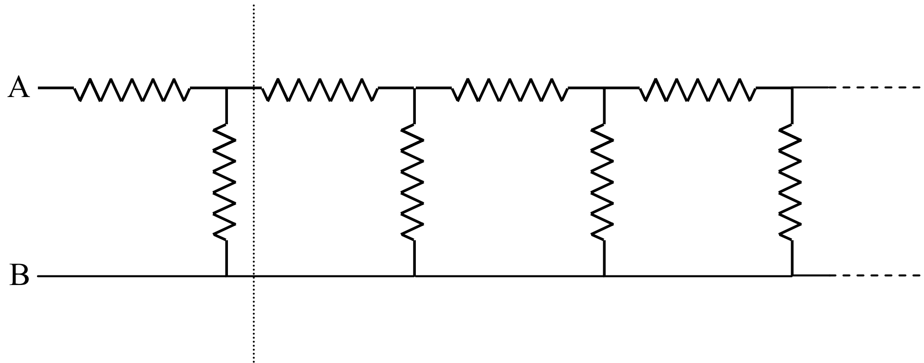


The figure shows six resistors, whose resistances in ohms are marked, arranged along the edges of a tetrahedron. Calculate the net resistance between C and D.

4.14.4 $R_1 = 8\ \Omega$ and $R_2 = 0.5\ \Omega$ are connected across a battery. The rate at which heat is generated is the same whether they are connected in series or in parallel. What is the internal resistance r of the battery?

4.14.5 $R_1 = 0.25\ \Omega$ and $R_2 = ?$ are connected across a battery whose internal resistance r is 0.5 Ω. The rate at which heat is generated is the same whether they are connected in series or in parallel. What is the value of R_2?

4.14.6



In the above circuit, each resistance is 1 ohm. What is the net resistance between A and B if the chain is of infinite length?

4.14.7 What is the resistance between A and B in question 4.14.6 if the chain is not of infinite length, but has n "links" – i.e. $2n$ resistors in all?

4.15 *Solutions, Answers or Hints to 4.14*

Hints for 4.14.1. Imagine a current of $6I$ going into the bottom left hand corner. Follow the current through the cube, writing down the current through each of the 12 resistors. Also write down the potential drop across each resistor, and hence the total potential drop across the cube. I make the answer for the effective resistance of the whole cube $\frac{5}{6}r$.

Solution for 4.14.2. By symmetry, the potentials of A and B are equal. Therefore there is no current between A and B.

Hint for 4.14.3. Replace the heavily-drawn delta with its corresponding star. After that it should be straightforward, although there is a little bit of calculation to do. I make the answer $1.52\ \Omega$.

Solution for 4.14. From equation 4.8.1, the rate at which heat is generated in a resistance R connected across a battery of EMF E and internal resistance r is $\dfrac{E^2R}{(R+r)^2}$.

If the resistors are connected in series, $R = R_1 + R_2$, while if they are connected in parallel, $R = \dfrac{R_1 R_2}{R_1 + R_2}$. If the heat generated is the same in either case, we must have

$$\frac{R_1 + R_2}{(R_1 + R_2 + r)^2} = \frac{\frac{R_1 R_2}{R_1 + R_2}}{\left(\frac{R_1 R_2}{R_1 + R_2} + r\right)^2}.$$

After some algebra, we obtain

$$r = \frac{R_1 + R_2 - \sqrt{R_1 R_2}}{\sqrt{\frac{R_1}{R_2}} + \sqrt{\frac{R_2}{R_1}} - 1}. \qquad 4.15.1$$

With $R_1 = 8\ \Omega$ and $R_2 = 0.5\ \Omega$, we obtain $\underline{r = 2.00\ \Omega\ .}$

Solution for 4.14.5.

In equation 4.15.1, let $\frac{r}{R_1} = a$ and $\sqrt{\frac{R_2}{R_1}} = x$. The equation 4.15.1 becomes

$$a = \frac{1 + x^2 - x}{(1/x) + x - 1}. \qquad 4.15.2$$

Upon rearrangement, this is

$$a - (a+1)x + (a+1)x^2 - x^3 = 0. \qquad 4.15.3$$

In our example, $a = \frac{r}{R_1} = \frac{0.50}{0.25} = 2$, so that equation 4.15.4 is

$$2 - 3x + 3x^2 - x^3 = 0,$$

or

$$(2 - x)(1 - x + x^2) = 0.$$

The only real root is $x = 2$. But $R_2 = R_1 x^2 = 0.25x^2 = 1\Omega$.

Solution to 4.14.6 Suppose that there are n links ($2n$ resistors) to the right of the dotted line, and that the effective resistance of these n links is R_n. Add one more link, to the left. The effective resistance of the $n + 1$ links is then

$$R_{n+1} = \frac{2R_n + 1}{R_n + 1}. \qquad 4.15.4$$

As $n \to \infty$, $R_{n+1} \to R_n \to R$. $\therefore\ R = \dfrac{2R+1}{R+1}$, or $R^2 - R - 1 = 0$.

Whence, $R = \frac{1}{2}(\sqrt{5} + 1) = 1.618\ 033\ 989\ \Omega$.

Solution to 4.14.7 By repeated application of equation 4.15.4, we find:

n	R_n	
1	2	
2	$\frac{5}{3}$	= 1.666 666 667
3	$\frac{13}{8}$	= 1.625 000 000
4	$\frac{34}{21}$	= 1.619 047 619
5	$\frac{89}{55}$	= 1.618 181 818
6	$\frac{233}{144}$	= 1.618 055 556
7	$\frac{610}{377}$	= 1.618 037 135
8	$\frac{1597}{987}$	= 1.618 034 448
9	$\frac{4181}{2584}$	= 1.618 034 056
10	$\frac{10946}{6765}$	= 1.618 033 999
11	$\frac{28657}{17711}$	= 1.618 033 990
12	$\frac{75025}{46368}$	= 1.618 033 989

Inspection shows that $R_n = \dfrac{F_{2n+1}}{F_{2n}}$, where F_m is the mth member of the Fibonacci sequence: 1 1 2 3 5 8 13 21 ......

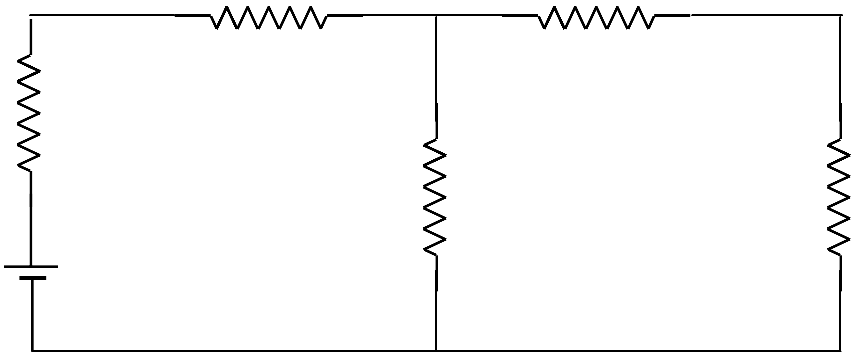
But, from the theory of Fibonacci sequences,

$$F_m = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right\}.$$

Hence
$$R_n = \frac{1}{2}\left(\frac{(1+\sqrt{5})^{2n+1} - (1-\sqrt{5})^{2n+1}}{(1+\sqrt{5})^{2n} - (1-\sqrt{5})^{2n}}\right)\ \Omega$$

4.16 *Attenuators*

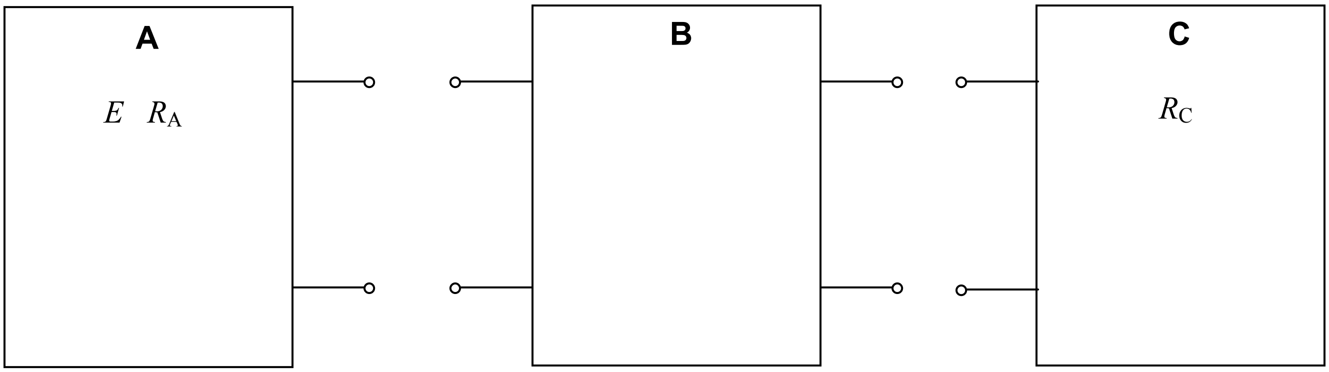
These are networks, usually of resistors, that serve the dual purpose of supplying more examples for students or for reducing the voltage, current or power from one circuit to another. An example of the former might be:



You might be told the values of the four left-hand resistances and of the EMF of the cell, and you are asked to find the current in the right hand resistor.

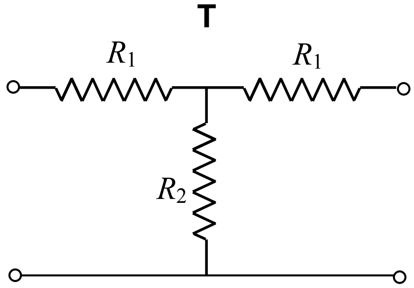
On the other hand, if the object is to design an attenuator, you might be told the values of the resistances at the two ends, and you are required to find the resistances of the three middle resistors such that the current in the rightmost resistor is half the current in the left hand resistor. The three intermediate resistances perform the function of an *attenuator.*

In the drawing below, **A** is some sort of a device, or electrical circuit, or, in the simplest case, just a battery, which has an electromotive force E and an internal resistance $R_{\rm A}$. **C** is some other device, whose internal resistance is $R_{\rm C}$. **B** is an attenuator, which is a collection of resistors which you want to design so that the current delivered to **C** is a certain fraction of the current flowing from **A**; or so that the voltage delivered across the terminals of **C** is a certain fraction of the voltage across the terminals of **A**; or perhaps again so that the power delivered to **C** is a certain fraction of the power generated by **A**. The circuit in the attenuator has to be designed so as to achieve one of these goals.

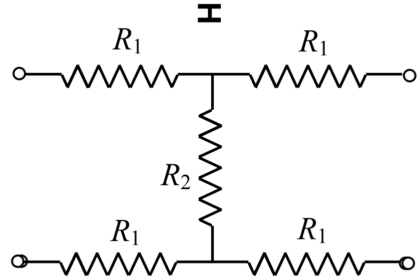


Four simple attenuators are known as **T**, **H,** Π or square, named after their shapes. In the drawing below, the **H** is on its side, like this: **H**

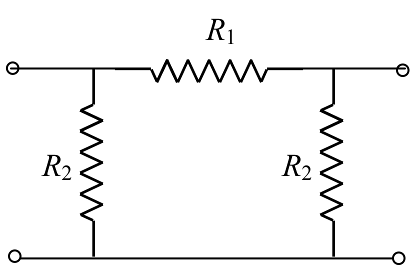
T



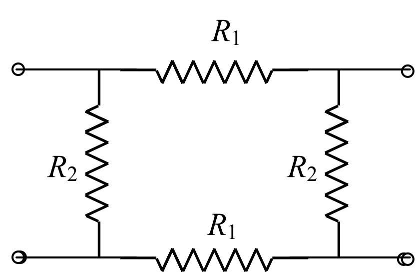
H



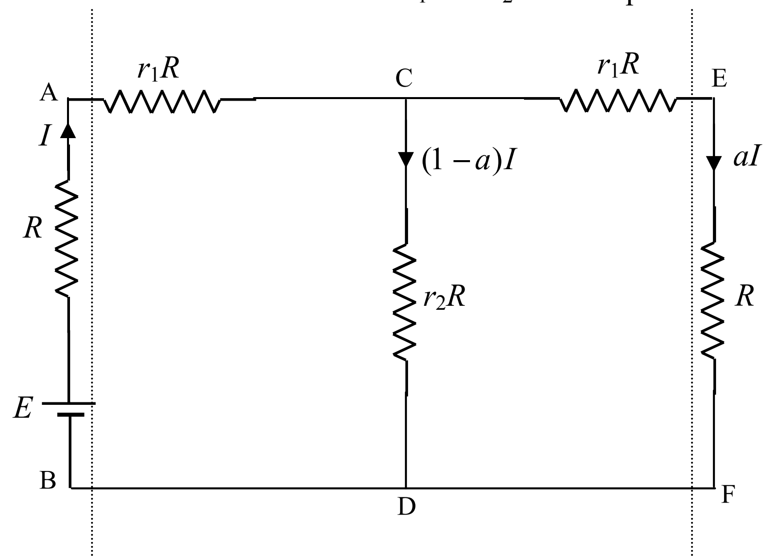
Π



□



Let us look at a **T** attenuator. We'll suppose that the **A** device has an electromotive force E and an internal resistance R, and indeed it can be represented by a cell in series with a resistor. And we'll suppose that the resistance of the **C** device is also R and it can be represented by a single resistor. We'll suppose that we want the current that flows into **C** to be a fraction a of the current flowing out of **A**, and the voltage to be supplied to **C** to be a fraction a of the potential difference across the output terminals of **A**. What must be the values of the resistances in the **T** attenuator? I'll call them r_1R and r_2R, so that we have to determine the dimensionless ratios r_1 and r_2. The equivalent circuit is:



The current leaving the battery is I, and we want the current entering the load at the right hand side to be aI. The current down the middle resistor is then necessarily, by Kirchhoff's first rule, $(1-a)I$. If we apply Kirchhoff's second rule to the outermost circuit, we obtain (after algebraic reduction)

$$E = (1+a)(1+r_1)IR. \qquad 4.16.1$$

We also want the potential difference across the load (i.e. across EF), which, by Ohm's law, is aIR, to be a times the potential difference across the source AB. AB are the terminals of the source. Recall that E is the EMF of the source, and R its internal resistance, so that, when a current I is being taken from the source, the potential difference across its terminals AB is $E - IR$, and we want aIR to be a times this. The fraction a can be called the *voltage reduction factor* of the attenuator. Thus we have

$$aIR = a(E - IR). \qquad 4.16.2$$

From these two equations we obtain

$$\underline{\underline{r_1 = \frac{1-a}{1+a}}}. \qquad 4.16.3$$

Application of Kirchhoff's second rule to the right hand circuit gives

$$r_2(1-a)IR \;=\; a(1+r_1)IR, \qquad 4.16.4$$

which, in combination with equation 4.16.3, yields

$$\underline{\underline{r_2 \;=\; \frac{2a}{1-a^2}}}\,. \qquad 4.16.5$$

Thus, if source and load resistance are each equal to R, and we want a voltage reduction factor of $\frac{1}{2}$, we must choose r_1R to be $\frac{1}{3}R$, and r_2R to be $\frac{4}{3}R$.

You might like to try the same problem for the ⌶, Π and square attenuators. I am not absolutely certain (I haven't checked them carefully), but I believe the answers are:

For ⌶ :

$$r_1 \;=\; \frac{1}{2}\left(\frac{1-a}{1+a}\right) \qquad\qquad r_2 \;=\; \frac{2a}{1-a^2} \qquad 4.16.6$$

For Π :

$$r_1 \;=\; \frac{1-a^2}{2a} \qquad\qquad r_2 \;=\; \frac{1+a}{1-a} \qquad 4.16.7$$

For square :

$$r_1 \;=\; \frac{1}{2}\left(\frac{1-a^2}{2a}\right) \qquad\qquad r_2 \;=\; \frac{1+a}{1-a} \qquad 4.16.7$$