CHAPTER 13
ALTERNATING CURRENTS

13.1 *Alternating current in an inductance*

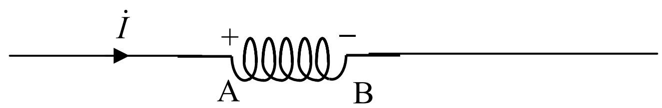


FIGURE XIII.1

In the figure we see a current increasing to the right and passing through an inductor. As a consequence of the inductance, a back EMF will be induced, with the signs as indicated. I denote the back EMF by $V = V_A - V_B$. The back EMF is given by $V = L\dot{I}$.

Now suppose that the current is an alternating current given by

$$I = \hat{I}\sin\omega t. \qquad 13.1.1$$

In that case $\dot{I} = \hat{I}\omega\cos\omega t$, and therefore the back EMF is

$$V = \hat{I}L\omega\cos\omega t, \qquad 13.1.2$$

which can be written

$$V = \hat{V}\cos\omega t, \qquad 13.1.3$$

where the peak voltage is

$$\hat{V} = L\omega\hat{I} \qquad 13.1.4$$

and, of course $V_{\text{RMS}} = L\omega I_{\text{RMS}}$.

The quantity $L\omega$ is called the *inductive reactance* X_L. It is expressed in ohms (check the dimensions), and, the higher the frequency, the greater the reactance. (The frequency ν is $\omega/(2\pi)$.)

Comparison of equations 13.1.1 and 13.1.3 shows that the current and voltage are out of phase, and that V leads on I by 90°, as shown in figure XIII.2.

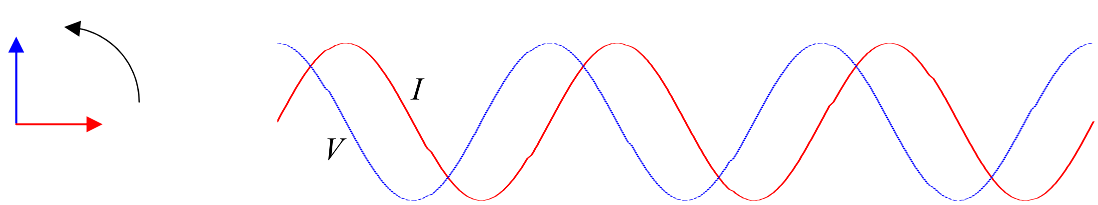


FIGURE XIII.2

13.2 *Alternating Voltage across a Capacitor*

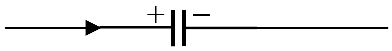


FIGURE XIII.3

At any time, the charge Q on the capacitor is related to the potential difference V across it by $Q = CV$. If there is a current in the circuit, then Q is changing, and $I = C\dot{V}$.

Now suppose that an alternating voltage given by

$$V = \hat{V}\sin\omega t \qquad 13.2.1$$

is applied across the capacitor.

In that case the current is $$I = C\omega\hat{V}\cos\omega t, \qquad 13.2.2$$

which can be written $$I = \hat{I}\cos\omega t, \qquad 13.2.3$$

where the peak current is $$\hat{I} = C\omega\hat{V} \qquad 13.2.4$$

and, of course $I_{\mathrm{RMS}} = C\omega V_{\mathrm{RMS}}$.

The quantity $1/(C\omega)$ is called the *capacitive reactance* X_C. It is expressed in ohms (check the dimensions), and, the higher the frequency, the smaller the reactance. (The frequency ν is $\omega/(2\pi)$.)

Comparison of equations 13.2.1 and 13.2.3 shows that the current and voltage are out of phase, and that V lags behind I by 90°, as shown in figure XIII.4.

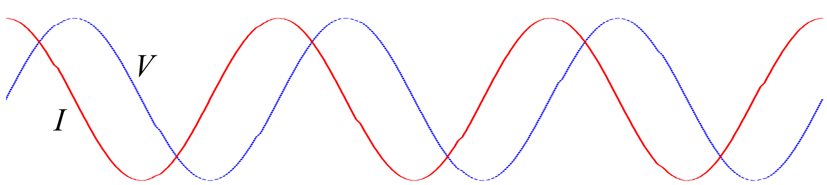


FIGURE XIII.4

13.3 *Complex Numbers*

I am now going to repeat the analyses of Sections 13.1 and 13.2 using the notation of complex numbers. In the context of alternating current theory, the imaginary unit is customarily given the symbol j rather than i, so that the symbol i is available, if need be, for electric currents. I am making the assumption that the reader is familiar with the basics of complex numbers; without that background, the reader may have difficulty with much of this chapter.

We start with the inductance. If the current is changing, there will be a back EMF given by $V = L\dot{I}$. If the current is changing as

$$I = \hat{I}e^{j\omega t}, \qquad 13.3.1$$

then $\dot{I} = \hat{I}j\omega e^{j\omega t} = j\omega I$. Therefore the voltage is given by

$$V = jL\omega I. \qquad 13.3.2$$

The quantity $jL\omega$ is called the *impedance* of the inductor, and is j times its reactance. Equation 13.3.2 (in particular the operator j on the right hand side) tells us that V leads on I by 90°.

Now suppose that an alternating voltage is applied across a capacitor. The charge on the capacitor at any time is $Q = CV$, and the current is $I = C\dot{V}$. If the voltage is changing as

$$V = \hat{V}e^{j\omega t}, \qquad 13.3.3$$

then $\dot{V} = \hat{V}j\omega e^{j\omega t} = j\omega V$. Therefore the current is given by

$$I = jC\omega V. \qquad 13.3.4$$

That is to say

$$V = -\frac{j}{C\omega}I. \qquad 13.3.5$$

The quantity $-j/(C\omega)$ is called the *impedance* of the capacitor, and is $-j$ times its reactance. Equation 13.3.5 (in particular the operator $-j$ on the right hand side) tells us that V lags behind I by 90°.

In summary:

Inductor:	Reactance = $L\omega$.	Impedance = $jL\omega$.	V leads on I.
Capacitor:	Reactance = $1/(C\omega)$.	Impedance = $-j/(C\omega)$.	V lags behind I.

It may be that at this stage you haven't got a very clear idea of the distinction between reactance (symbol X) and impedance (symbol Z) other than that one seems to be j or $-j$ times the other. The next section deals with a slightly more complicated situation, namely a resistor and an inductor in series. (In practice, it may be one piece of equipment, such as a solenoid, that has both resistance and inductance.) Paradoxically, you may find it easier to understand the distinction between impedance and reactance from this more complicated situation.

13.4 *Resistance and Inductance in Series*

The impedance is just the sum of the resistance of the resistor and the impedance of the inductor:

$$Z = R + jL\omega. \qquad 13.4.1$$

Thus the impedance is a *complex number*, whose real part R is the resistance and whose imaginary part $L\omega$ is the reactance. For a pure resistance, the impedance is real, and V and I are in phase. For a pure inductance, the impedance is imaginary (reactive), and there is a 90° phase difference between V and I.

The voltage and current are related by

$$V = IZ = (R + jL\omega)I. \qquad 13.4.2$$

Those who are familiar with complex numbers will see that this means that V leads on I, not by 90°, but by the *argument* of the complex impedance, namely $\tan^{-1}(L\omega / R)$. Further the ratio of the peak (or RMS) voltage to the peak (or RMS) current is equal to the *modulus* of the impedance, namely $\sqrt{R^2 + L^2\omega^2}$.

13.5 *Resistance and Capacitance in Series*

Likewise the impedance of a resistance and a capacitance in series is

$$Z = R - j/(C\omega). \qquad 13.5.1$$

The voltage and current are related, as usual, by $V = IZ$. Equation 13.5.1 shows that the voltage lags behind the current by $\tan^{-1}[1/(RC\omega)]$, and that $\hat{V}/\hat{I} = \sqrt{R^2 + 1/(C\omega)^2}$.

13.6 *Admittance*

In general, the impedance of a circuit is partly resistive and partly reactive:

$$Z = R + jX. \qquad 13.6.2$$

The real part is the resistance, and the imaginary part is the reactance. The relation between V and I is $V = IZ$. If the circuit is purely resistive, V and I are in phase. If is it purely reactive, V and I differ in phase by 90°. The reactance may be partly inductive and partly capacitive, so that

$$Z = R + j(X_\mathrm{L} - X_\mathrm{C}). \qquad 13.6.3$$

Indeed we shall describe such a system in detail in the next section.

The reciprocal of the impedance Z is the *admittance*, Y.

Thus
$$Y = \frac{1}{Z} = \frac{1}{R + jX}. \qquad 13.6.4$$

And of course, since $V = IZ,\ I = VY.$

Whenever we see a complex (or a purely imaginary) number in the denominator of an expression, we always immediately multiply top and bottom by the complex conjugate, so equation 13.6.4 becomes

$$Y = \frac{Z^*}{|Z|^2} = \frac{R - jX}{R^2 + X^2}. \qquad 13.6.5$$

This can be written

$$Y = G + jB, \qquad 13.6.6$$

where the real part, G, is the *conductance*:

$$G = \frac{R}{R^2 + X^2}, \qquad 13.6.7$$

and the imaginary part, B, is the *susceptance*:

$$B = -\frac{X}{R^2 + X^2}. \qquad 13.6.8$$

The SI unit for admittance, conductance and susceptance is the *siemens* (or the "mho" in informal talk).

I leave it to the reader to show that

$$R = \frac{G}{G^2 + B^2} \qquad 13.6.9$$

and
$$X = -\frac{B}{G^2 + B^2}. \qquad 13.6.10$$

13.7 *The RLC Series Acceptor Circuit*

A resistance, inductance and a capacitance in series is called an "acceptor" circuit, presumably because, for some combination of the parameters, the magnitude of the inductance is a minimum, and so current is accepted most readily. We see in figure XIII.5 an alternating voltage $V = \hat{V}e^{j\omega t}$ applied across such an R, L and C.

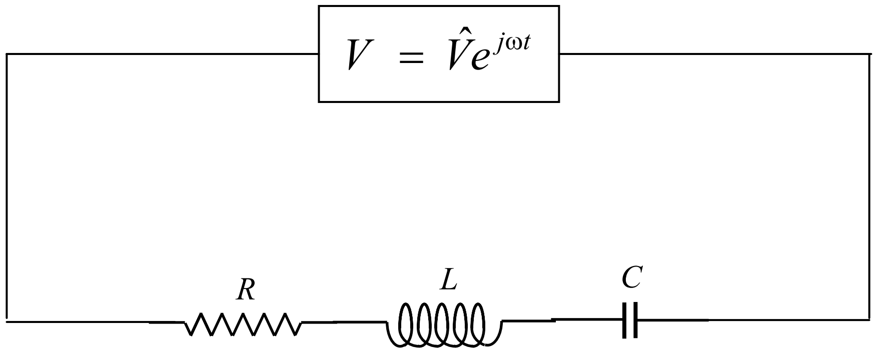


FIGURE XIII.5

The impedance is

$$Z = R + j\left(L\omega - \frac{1}{C\omega}\right). \qquad 13.7.1$$

We can see that the voltage leads on the current if the reactance is positive; that is, if the inductive reactance is greater than the capacitive reactance; that is, if $\omega > 1/\sqrt{LC}$. (Recall that the frequency, ν, is $\omega/(2\pi)$). If $\omega < 1/\sqrt{LC}$, the voltage lags behind the current. And if $\omega = 1/\sqrt{LC}$, the circuit is purely resistive, and voltage and current are in phase.

The magnitude of the impedance (which is equal to $\hat{V}/\hat{I}$) is

$$|Z| = \sqrt{R^2 + (L\omega - 1/(C\omega))^2}, \qquad 13.7.2$$

and this is least (and hence the current is greatest) when $\omega = 1/\sqrt{LC}$, the resonant frequency, which I shall denote by ω_0.

It is of interest to draw a graph of how the magnitude of the impedance varies with frequency for various values of the circuit parameters. I can reduce the number of parameters by defining the dimensionless quantities

$$\Omega = \omega/\omega_0 \qquad 13.7.3$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad 13.7.4$$

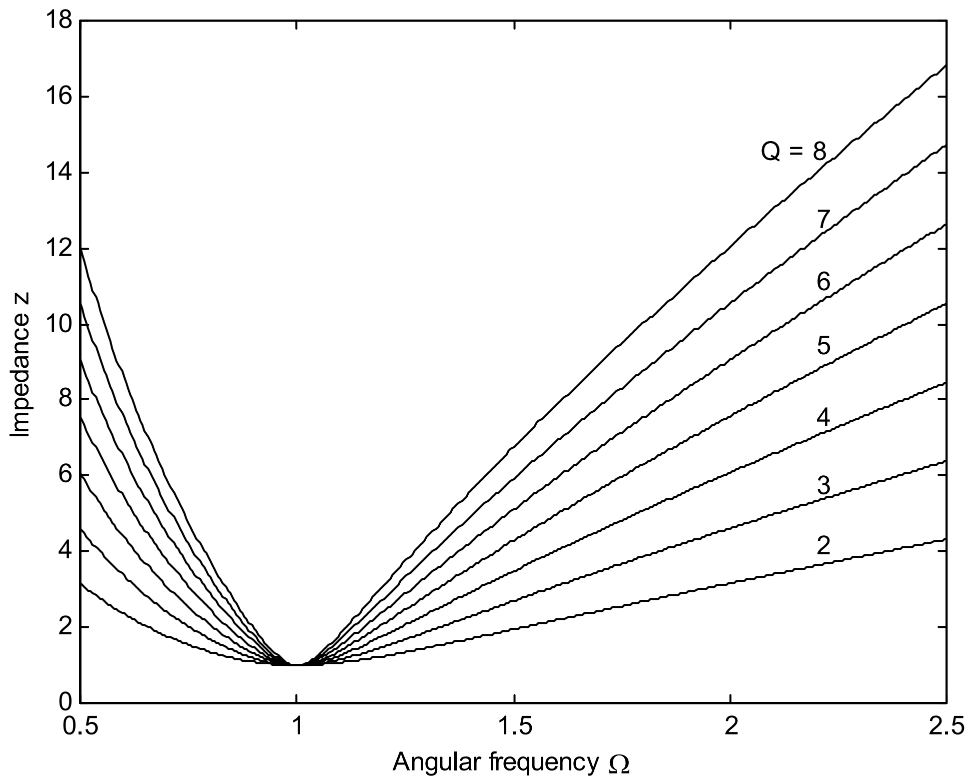
and $$z = \frac{|Z|}{R}. \qquad 13.7.4$$

You should verify that Q is indeed dimensionless. We shall see that the sharpness of the resonance depends on Q, which is known as the *quality factor* (hence the symbol Q). In terms of the dimensionless parameters, equation 13.7.2 becomes

$$z = \sqrt{1 + Q^2(\Omega - 1/\Omega)^2}. \qquad 13.7.5$$

This is shown in figure XIII.6, in which it can be seen that the higher the quality factor, the sharper the resonance.

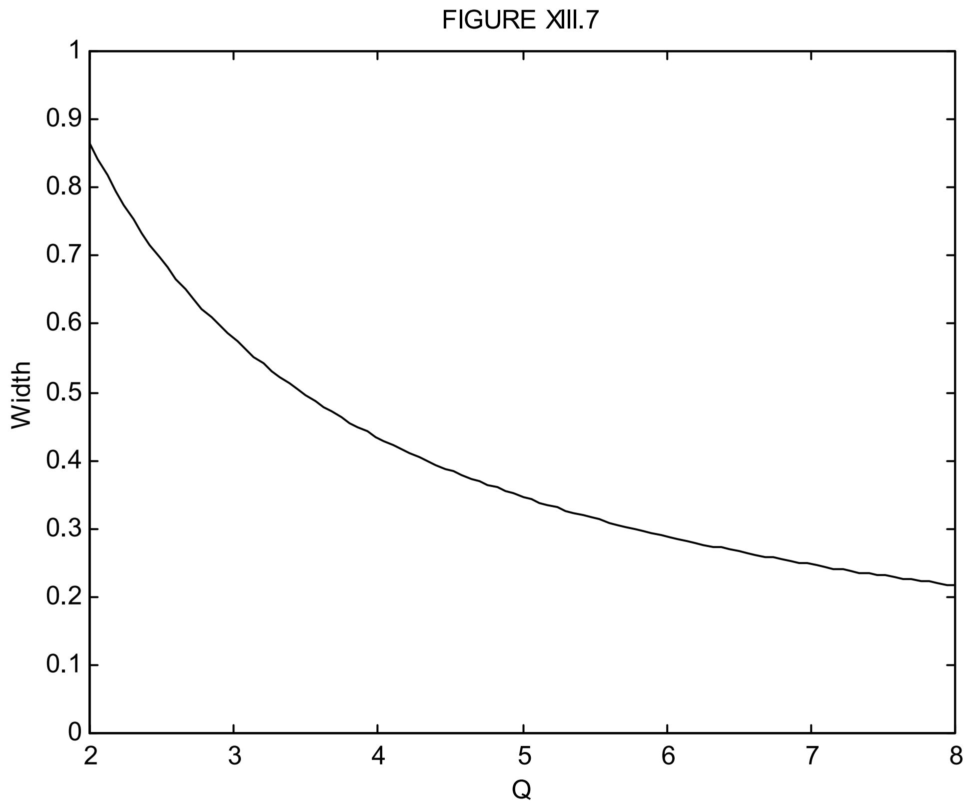
FIGURE XIII.6



In particular, it is easy to show that the frequencies at which the impedance is twice its minimum value are given by the positive solutions of

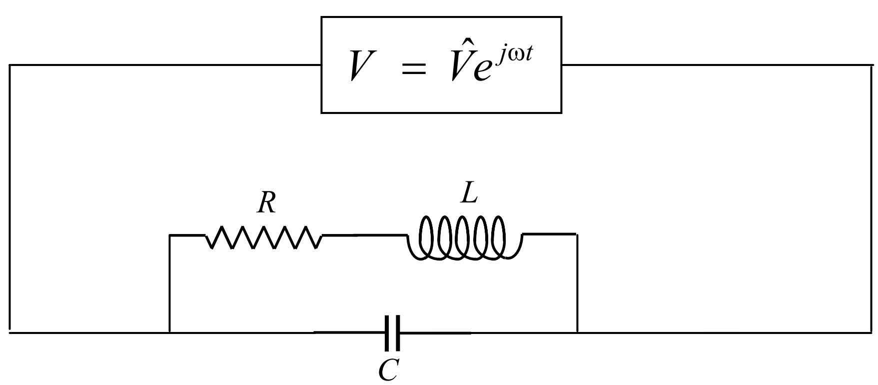
$$\Omega^4 - \left(2 + \frac{3}{Q^2}\right)\Omega^2 + 1 = 0. \qquad 13.7.6$$

If I denote the smaller and larger of these solutions by Ω_- and Ω_+, then $\Omega_+ - \Omega_-$ will serve as a useful description of the width of the resonance, and this is shown as a function of quality factor in figure XIII.7.



13.8 *The RLC Parallel Rejector Circuit*

In the circuit below, the magnitude of the *admittance* is least for certain values of the parameters. When you tune a radio set, you are changing the overlap area (and hence the capacitance) of the plates of a variable air-spaced capacitor so that the admittance is a minimum for a given frequency, so as to ensure the highest potential difference across the circuit. This resonance, as we shall see, does not occur for an angular frequency of exactly $1/\sqrt{LC}$, but at an angular frequency that is approximately this if the resistance is small.



The admittance is

$$Y = jC\omega + \frac{1}{R + jL\omega}. \qquad 13.8.1$$

After some routine algebra (multiply top and bottom by the conjugate; then collect real and imaginary parts), this becomes

$$Y = \frac{R + j\omega(L^2C\omega^2 + R^2C - L)}{R^2 + L^2\omega^2}. \qquad 13.8.2$$

The magnitude of the admittance is least when the susceptance is zero, which occurs at an angular frequency of

$$\omega_0^2 = \frac{1}{LC} - \frac{R^2}{L^2}. \qquad 13.8.3$$

If $R << \sqrt{L/C}$, this is approximately $1/\sqrt{LC}$.

13.9 *AC Bridges*

We have already met, in Chapter 4, Section 4.11, the Wheatstone bridge, which is a DC (direct current) bridge for comparing resistances, or for "measuring" an unknown resistance if it is compared with a known resistance. In the Wheatstone bridge (figure IV.9), balance is achieved when $\frac{R_1}{R_2} = \frac{R_3}{R_4}$. Likewise in a AC (alternating current) bridge, in which the power supply is an AC generator, and there are impedances (combinations of *R*, *L* and *C*) in each arm (figure XIII.8),

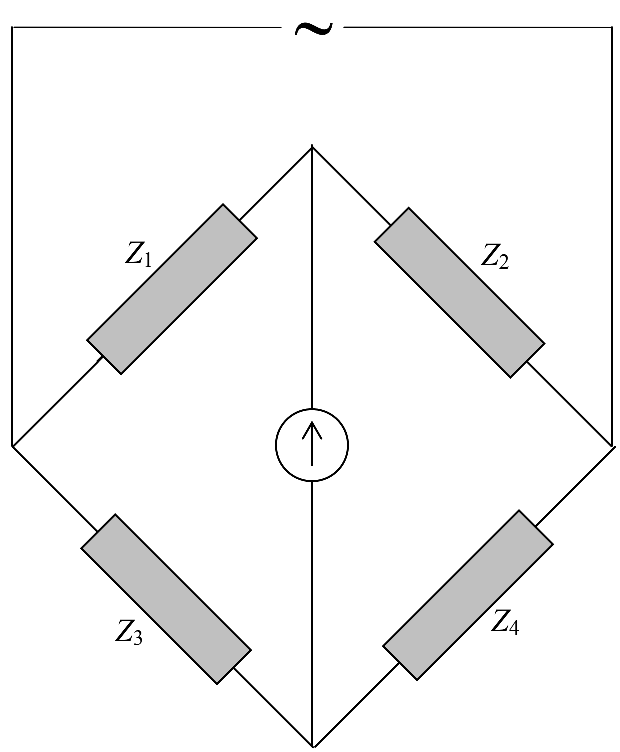


FIGURE XIII.8

balance is achieved when

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4} \tag{13.9.1}$$

or, of course, $\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}$. This means not only that the RMS potentials on both sides of the detector must be equal, but they must be *in phase*, so that the potentials are the same *at all times.* (I have drawn the "detector" as though it were a galvanometer, simply because that is easiest for me to draw. In practice, it might be a pair of earphones or an oscilloscope.) Each side of equation 13.9.1 is a complex number, and two complex numbers are equal if and only if their real and imaginary parts are separately equal. Thus equation 13.9.1 really represents two equations – which are necessary in order to satisfy the two conditions that the potentials on either side of the detector are equal in magnitude and in phase.

We shall look at three examples of AC bridges. It is not recommended that these be committed to memory. They are described only as examples of how to do the calculation.

13 9.1 *The Owen Bridge*

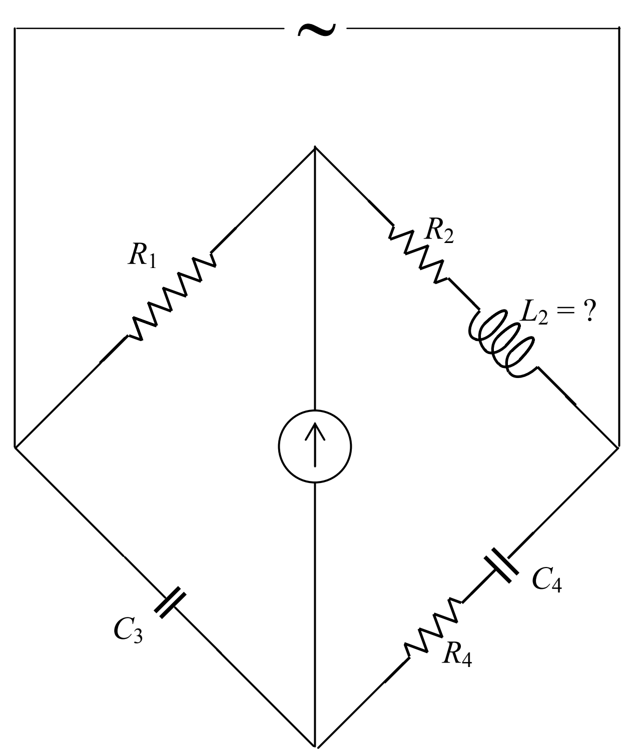


FIGURE XIII.9

This bridge can be used for measuring inductance. Note that the unknown inductance is the only inductance in the bridge. Reactance is supplied by the capacitors.

Equation 13.9.1 in this case becomes

$$\frac{R_1}{R_2 + jL_2\omega} = \frac{-j/(C_3\omega)}{R_4 - j/(C_4\omega)}. \tag{13.9.2}$$

That is,

$$R_1R_4 - j\frac{R_1}{C_4\omega} = \frac{L_2}{C_3} - j\frac{R_2}{C_3\omega}. \tag{13.9.3}$$

On equating real and imaginary parts separately, we obtain

$$L_2 = R_1R_4C_3 \tag{13.9.4}$$

and

$$\frac{R_1}{R_2} = \frac{C_4}{C_3}. \tag{13.9.5}$$

13 9.2 *The Schering Bridge*

This bridge can be used for measuring capacitance.

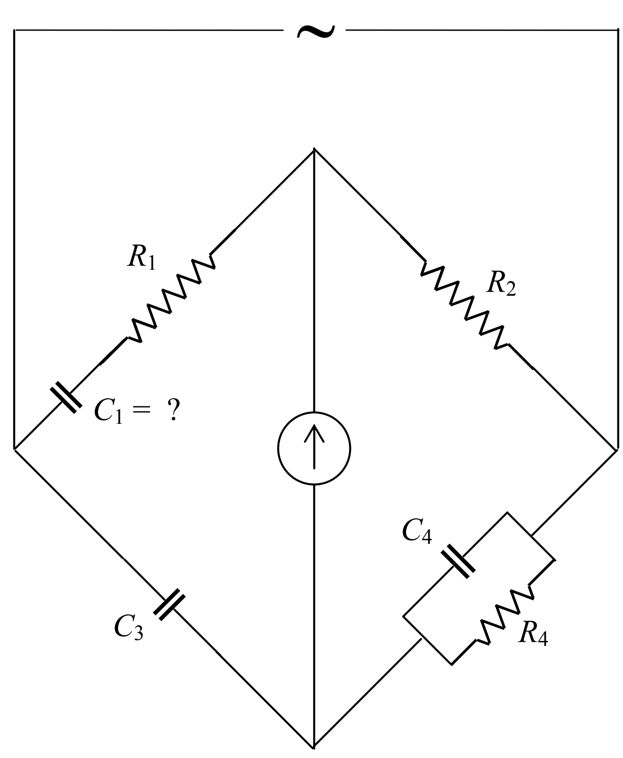


FIGURE XIII.10

The admittance of the fourth arm is $\frac{1}{R_4} + jC_4\omega$, and its impedance is the reciprocal of this. I leave the reader to balance the bridge and to show that

$$\frac{R_1}{R_2} = \frac{C_4}{C_3} \qquad 13.9.6$$

and

$$C_1 = \frac{C_3 R_4}{R_2}. \qquad 13.9.7$$

13 9.3 *The Wien Bridge*

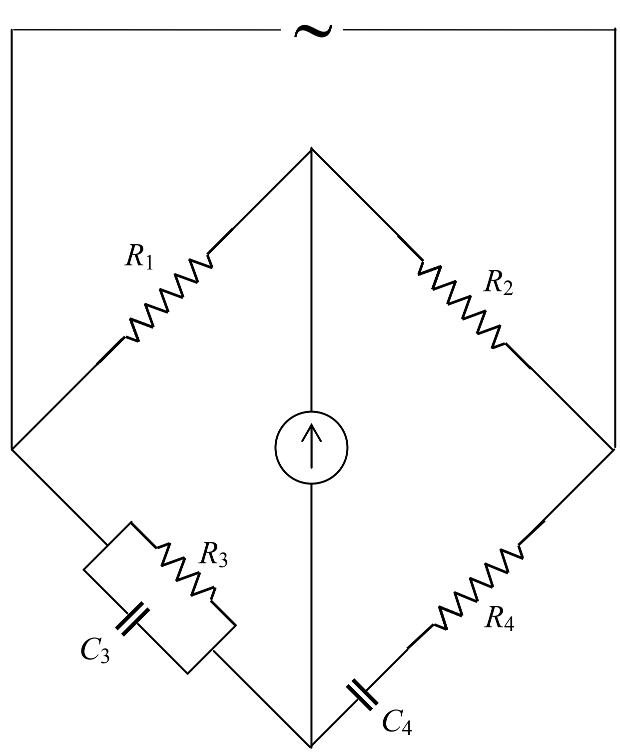


FIGURE XIII.11

This bridge can be used for measuring frequency.

The reader will, I think, be able to show that

$$\frac{R_4}{R_3} + \frac{C_3}{C_4} = \frac{R_2}{R_1} \qquad 13.9.8$$

and
$$\omega^2 = \frac{1}{R_3R_4C_3C_4} \,. \qquad 13.9.10$$

13.10 *The Transformer*

We met the transformer briefly in Section 10.9. There we pointed out that the EMF induced in the secondary coil is equal to the number of turns in the secondary coil times the rate of change of magnetic flux; and the flux is proportional to the EMF applied to the primary times the number of turns in the primary. Hence we deduced the well known relation

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \qquad 13.10.1$$

relating the primary and secondary voltages to the number of turns in each. We now look at the transformer in more detail; in particular, we look at what happens when we connect the secondary coil to a circuit and take power from it.

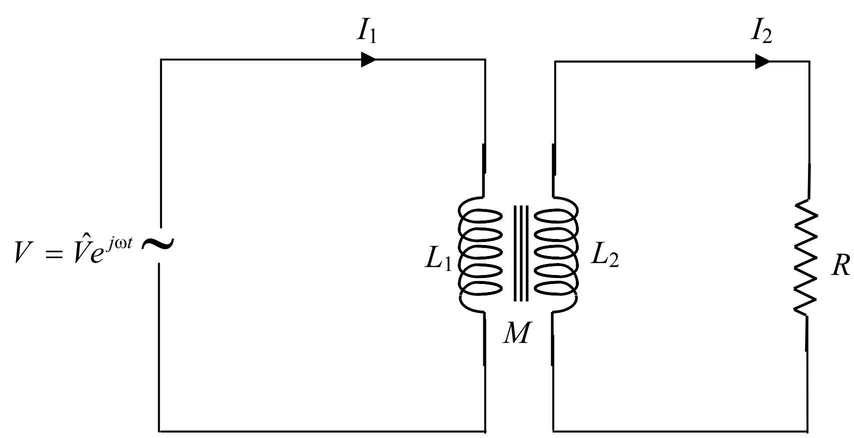


FIGURE XIII.12

In figure XIII.12, we apply an AC EMF $V = \hat{V}e^{j\omega t}$ to the primary circuit. The self inductance of the primary coil is L_1, and an alternating current I_1 flows in the primary circuit. The self inductance of the secondary coil is L_2, and the mutual inductance of the two coils is M. If the coupling between the two coils is very tight, then $M = \sqrt{L_1L_2}$; otherwise it is less than this. I am supposing that the resistance of the primary circuit is much smaller than the reactance, so I am going to neglect it.

The secondary coil is connected to a resistance R. An alternating current I_2 flows in the secondary circuit.

Let us apply Ohm's law (or Kirchhoff's second rule) to each of the two circuits.

In the *primary* circuit, the applied EMF V is opposed by two back EMF's:

$$V = L_1\dot{I}_1 + M\dot{I}_2. \qquad 13.10.2$$

That is to say

$$V = j\omega L_1 I_1 + j\omega M I_2. \qquad 13.10.3$$

Similarly for the *secondary* circuit:

$$0 = j\omega M I_1 + j\omega L_2 I_2 + R I_2. \qquad 13.10.4$$

These are two simultaneous equations for the currents, and we can (with a small effort) solve them for I_1 and I_2:

$$\left[\frac{RL_1}{M} + j\left(\frac{\omega L_1 L_2}{M} - \omega M\right)\right]I_1 = \left(\frac{L_2}{M} - j\frac{R}{\omega M}\right)V \qquad 13.10.5$$

and

$$\left[R + j\left(\omega L_2 - \frac{\omega M^2}{L_1}\right)\right]I_2 = -\frac{MV}{L_1}. \qquad 13.10.6$$

This would be easier to understand if we were to do the necessary algebra to write these in the forms $I_1 = (a + jb)V$ and $I_2 = (c + jd)V$. We could then easily see the phase relationships between the current and V as well as the peak values of the currents. There is no reason why we should not try this, but I am going to be a bit lazy before I do it, and I am going to assume that we have a well designed transformer in which the secondary coil is really tightly wound around the primary, and $M = \sqrt{L_1 L_2}$. If you wish, you may carry on with a less efficient transformer, with $M = k\sqrt{L_1 L_2}$, where k is a coupling coefficient less than 1, but I'm going to stick with $M = \sqrt{L_1 L_2}$. In that case, equations 13.10.5 and 6 eventually take the forms

$$I_1 = \left(\frac{L_2}{L_1 R} - j\frac{1}{L_1\omega}\right)V = \left(\frac{N_2^2}{N_1^2 R} - j\frac{1}{L_1\omega}\right)V \qquad 13.10.7$$

and

$$I_2 = -\frac{1}{R}\sqrt{\frac{L_2}{L_1}}V = -\frac{N_2}{N_1 R}V. \qquad 13.10.8$$

These equations will tell us, on examination, the magnitudes of the currents, and their phases relative to V.

Now look at the circuit shown in figure XIII.13.

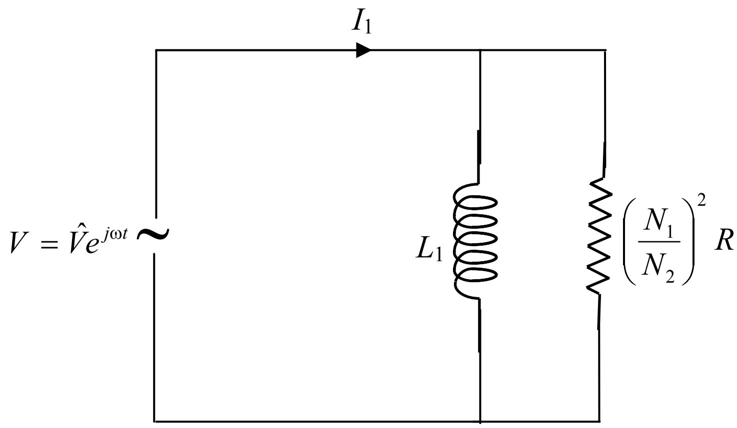


FIGURE XIII.13

In figure XIII.13 we have a resistance $R(N_1 / N_2)^2$ in parallel with an inductance L_1. The admittances of these two elements are, respectively, $(N_2 / N_1)^2 / R$ and $-j / (L_1\omega)$, so the total admittance is $\frac{N_2^2}{N_1^2 R} - j\frac{1}{L_1\omega}\cdot$ Thus, as far as the relationship between current and voltage is concerned, the primary circuit of the transformer is precisely equivalent to the circuit drawn in figure XIII.13. To see the relationship between I_1 and V, we need look no further than figure XIII.13.

Likewise, equation 13.10.8 shows us that the relationship between I_2 and V is exactly as if we had an AC generator of EMF $N_2 E / N_1$ connected across R, as in figure XIII.14.

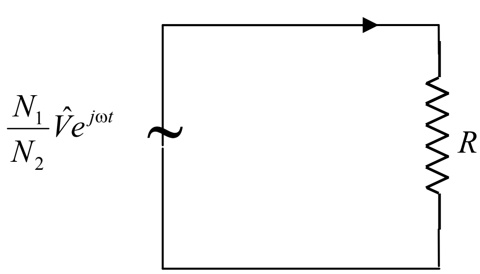


FIGURE XIII.14

Note that, if the secondary is short-circuited (i.e. if $R = 0$ and if the resistance of the secondary coil is literally zero) both the primary and secondary current become infinite. If the secondary circuit is left open (i.e. $R = \infty$), the secondary current is zero (as expected), and the primary current, also as expected, is not zero but is $-jV/(L_1\omega)$; That is to say, the current is of magnitude $V/(L_1\omega)$ and it lags behind the voltage by 90°, just as if the secondary circuit were not there.