CHAPTER 15
MAXWELL'S EQUATIONS

15.1 *Introduction*

One of Newton's great achievements was to show that all of the phenomena of classical mechanics can be deduced as consequences of three basic, fundamental laws, namely Newton's laws of motion. It was likewise one of Maxwell's great achievements to show that all of the phenomena of classical electricity and magnetism – all of the phenomena discovered by Oersted, Ampère, Henry, Faraday and others whose names are commemorated in several electrical units – can be deduced as consequences of four basic, fundamental equations. We describe these four equations in this chapter, and, in passing, we also mention Poisson's and Laplace's equations. We also show how Maxwell's equations predict the existence of electromagnetic waves that travel at a speed of 3×10^8 m s^{-1}. This is the speed at which light is measured to move, and one of the most important bases of our belief that light is an electromagnetic wave.

Before embarking upon this, we may need a reminder of two mathematical theorems, as well as a reminder of the differential equation that describes wave motion.

The two mathematical theorems that we need to remind ourselves of are:

The surface integral of a vector field over a closed surface is equal to the volume integral of its divergence.

The line integral of a vector field around a closed plane curve is equal to the surface integral of its curl.

A function $f(x - \upsilon t)$ represents a function that is moving with speed υ in the positive x-direction, and a function $g(x + \upsilon t)$ represents a function that is moving with speed υ in the negative x-direction. It is easy to verify by substitution that $y = Af + Bg$ is a solution of the differential equation

$$\frac{d^2 y}{dt^2} = \upsilon^2 \frac{d^2 y}{dx^2}. \qquad 15.1.1$$

Indeed it is the most general solution, since f and g are quite general functions, and the function y already contains the only two arbitrary integration constants to be expected from a second order differential equation. Equation 15.1.1 is, then, the differential equation for a wave in one dimension. For a function $\psi(x, y, z)$ in three dimensions, the corresponding wave equation is

$$\ddot{\psi} = \upsilon^2 \nabla^2 \psi. \qquad 15.1.2$$

It is easy to remember which side of the equation υ^2 is on from dimensional considerations.

One last small point before proceeding – I may be running out of symbols! I may need to refer to *surface charge density*, a scalar quantity for which the usual symbol is σ. I shall also need to refer to *magnetic vector potential*, for which the usual symbol is $\mathbf{A}$. And I shall need to refer to *area*, for which either of the symbols A or σ are commonly used – or, if the vector nature of area is to be emphasized, $\mathbf{A}$ or $\boldsymbol{\sigma}$. What I shall try to do, then, to avoid this difficulty, is to use $\mathbf{A}$ for magnetic vector potential, and $\boldsymbol{\sigma}$ for area, and I shall try to avoid using surface charge density in any equation. However, the reader is warned to be on the lookout and to be sure what each symbol means in a particular context.

15.2 *Maxwell's First Equation*

Maxwell's first equation, which describes the electrostatic field, is derived immediately from Gauss's theorem, which in turn is a consequence of Coulomb's inverse square law. Gauss's theorem states that the surface integral of the electrostatic field $\mathbf{D}$ over a closed surface is equal to the charge enclosed by that surface. That is

$$\int_{\text{surface}} \mathbf{D} \cdot \mathbf{d}\boldsymbol{\sigma} = \int_{\text{volume}} \rho \, dv. \qquad 15.2.1$$

Here ρ is the charge per unit volume.

But the surface integral of a vector field over a closed surface is equal to the volume integral of its divergence, and therefore

$$\int_{\text{volume}} \text{div}\mathbf{D} \, dv = \int_{\text{volume}} \rho \, dv. \qquad 15.2.2$$

Therefore $$\text{div } \mathbf{D} = \rho , \qquad 15.2.3$$

or, in the nabla notation, $$\nabla \cdot \mathbf{D} = \rho. \qquad 15.2.4$$

This is the first of Maxwell's equations.

15.3 *Poisson's and Laplace's Equations*

Equation 15.2.4 can be written $\nabla \cdot \mathbf{E} = \rho / \varepsilon$. where ε is the permittivity. But $\mathbf{E}$ is minus the potential gradient; i.e. $\mathbf{E} = -\nabla V$. Therefore,

$$\nabla^2 V = -\rho / \varepsilon. \qquad 15.3.1$$

This is *Poisson's equation.* At a point in space where the charge density is zero, it becomes

$$\nabla^2 V = 0, \qquad 15.3.2$$

which is generally known as *Laplace's equation.* Thus, regardless of how many charged bodies there may be an a place of interest, and regardless of their shape or size, the potential at any point can be calculated from Poisson's or Laplace's equations. Courses in differential equations commonly discuss how to solve these equations for a variety of *boundary conditions* – by which is meant the size, shape and location of the various charged bodies and the charge carried by each. It perhaps just needs to be emphasized that Poisson's and Laplace's equations apply only for *static* fields.

15.4 *Maxwell's Second Equation*

Unlike the electrostatic field, magnetic fields have no sources or sinks, and the magnetic lines of force are closed curves. Consequently the surface integral of the magnetic field over a closed surface is zero, and therefore

$$\text{div}\,\mathbf{B} = 0\,, \qquad 15.4.1$$

or, in the nabla notation

$$\nabla\cdot\mathbf{B} = 0. \qquad 15.4.2$$

This is the second of Maxwell's equations.

15.5 *Maxwell's Third Equation*

This is derived from Ampère's theorem, which is that the line integral of the magnetic field **H** around a closed circuit is equal to the enclosed current.

Now there are two possible components to the "enclosed" current, one of which is obvious, and the other, I suppose, could also be said to be "obvious" once it has been pointed out! Let's deal with the immediately obvious one first, and look at figure XV.1.

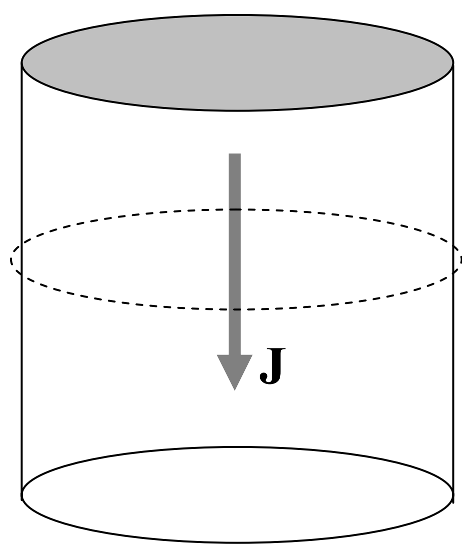

FIGURE XV.1

In figure XV.1, I am imagining a metal cylinder with current flowing from top to bottom (i.e. electrons flowing from bottom to top. It needn't be a metal cylinder, though. It could just be a volume of space with a stream of protons moving from top to bottom. In any case, the current density (which may vary with distance from the axis of the cylinder) is **J**, and the total current enclosed by the dashed circle is the integral of **J** throughout the cylinder. In a more general geometry, in which **J** is not necessarily perpendicular to the area of interest, and indeed in which the area need not be planar, this would be $\int \mathbf{J} \cdot \mathbf{d\sigma}$.

Now for the less obvious component to the "enclosed current". See figure XV.2.

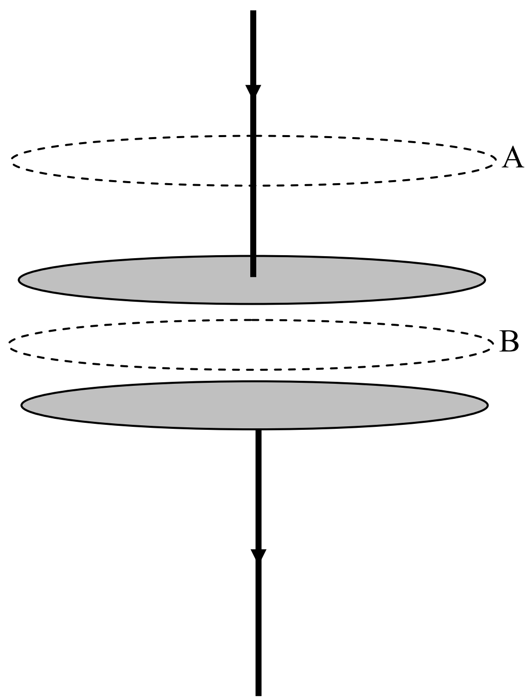

FIGURE XV.2

In figure XV.2, I imagine two capacitor plates in the process of being charged. There is undoubtedly a current flowing in the connecting wires. There is a magnetic field at A, and the line integral of the field around the upper dotted curve is undoubtedly equal to the enclosed current. The current is equal to the rate at which charge is being built up on the plates. Electrons are being deposited on the lower plate and are leaving the upper plate. There is also a magnetic field at B (it doesn't suddenly stop!), and the field at B is just the same as the field at A, which is equal to the rate at which charge is being built up on the plates. The charge on the plates (which may not be uniform, and indeed won't be while the current is still flowing or if the plates are not infinite in extent) is equal to the integral of the charge density times the area. And the charge density on the plates, by Gauss's theorem, is equal to the electric field **D** between the plates. Thus the current is equal to the integral of $\dot{\mathbf{D}}$ over the surface of the plates. Thus the line integral of **H** around either of the dashed closed loops is equal to $\int \dot{\mathbf{D}} \cdot \mathbf{d\sigma}$.

In general, both types of current (the obvious one in which there is an obvious flow of charge, and the less obvious one, where the electric field is varying because of a real flow of charge elsewhere) contributes to the magnetic field, and so Ampère's theorem in general must read

$$\int_{\text{loop}} \mathbf{H} \cdot \mathbf{ds} = \int_{\text{area}} (\dot{\mathbf{D}} + \mathbf{J}) \cdot \mathbf{d\sigma}. \qquad 15.5.1$$

But the line integral of a vector field around a closed plane curve is equal to the surface integral of its curl, and therefore

$$\int_{\text{area}} \mathbf{curl\ H} \cdot \mathbf{d\sigma} = \int_{\text{area}} (\dot{\mathbf{D}} + \mathbf{J}) \cdot \mathbf{d\sigma}. \qquad 15.5.2$$

Thus we arrive at:

$$\mathbf{curl\ H} = \dot{\mathbf{D}} + \mathbf{J}, \qquad 15.5.3$$

or, in the nabla notation,

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}. \qquad 15.5.4$$

This is the third of Maxwell's equations.

15.6 *The Magnetic Equivalent of Poisson's Equation*

This deals with a static magnetic field, where there is no electrostatic field or at least any electrostatic field is indeed static – i.e. not changing. In that case $\mathbf{curl\ H} = \mathbf{J}$. Now the magnetic field can be derived from the curl of the magnetic vector potential, defined by the two equations

$$\mathbf{B} = \mathbf{curl\ A} \qquad 15.6.1$$

and

$$\text{div}\ \mathbf{A} = 0. \qquad 15.6.2$$

(See Chapter 9 for a reminder of this.) Together with $\mathbf{H} = \mathbf{B}/\mu$ (μ = permeability), this gives us

$$\mathbf{curl\ curl\ A} = \mu \mathbf{J}. \qquad 15.6.3$$

If we now remind ourselves of the jabberwockian-sounding vector differential operator equivalence

$$\mathbf{curl\ curl} \equiv \mathbf{grad}\ \text{div} - \text{nabla-squared}, \qquad 15.6.4$$

together with equation 15.6.2, this gives us

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \qquad 15.6.5$$

I don't know if this equation has any particular name, but it plays the same role for static magnetic fields that Poisson's equation plays for electrostatic fields. No matter what the distribution of currents, the magnetic vector potential at any point must obey equation 15.6.5

15.7 *Maxwell's Fourth Equation*

This is derived from the laws of electromagnetic induction.

Faraday's and Lenz's laws of electromagnetic induction tell us that the E.M.F. induced in a closed circuit is equal to minus the rate of change of B-flux through the circuit. The E.M.F. around a closed circuit is the line integral of **E .ds** around the circuit, where **E** is the electric field. The line integral of **E** around the closed circuit is equal to the surface integral of its curl. The rate of change of B-flux through a circuit is the surface integral of $\dot{\mathbf{B}}$. Therefore

$$\mathbf{curl\,E} = -\dot{\mathbf{B}}, \tag{15.7.1}$$

or, in the nabla notation,

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}}. \tag{15.7.2}$$

This is the fourth of Maxwell's equations.

15.8 *Summary of Maxwell's and Poisson's Equations*

Maxwell's equations:

$$\nabla\bullet\mathbf{D}=\rho \tag{15.8.1}$$

$$\nabla\cdot\mathbf{B} = 0. \tag{15.8.2}$$

$$\nabla\times\mathbf{H} = \dot{\mathbf{D}} + \mathbf{J}. \tag{15.8.3}$$

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}}. \tag{15.8.4}$$

Sometimes you may see versions of these equations with factors such as 4π or c scattered liberally throughout them. If you do, my best advice is to white them out with a bottle of erasing fluid, or otherwise ignore them. I shall try to explain in Chapter 16 where they come from. They serve no scientific purpose, and are merely conversion factors between the many different systems of units that have been used in the past.

Poisson's equation for the potential in an electrostatic field:

$$\nabla^2 V = -\rho/\varepsilon. \tag{15.8.5}$$

The equivalent of Poisson's equation for the magnetic vector potential on a static magnetic field:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \qquad 15.8.6$$

15.9 *Electromagnetic Waves*

Maxwell predicted the existence of electromagnetic waves, and these were generated experimentally by Hertz shortly afterwards. In addition, the predicted speed of the waves was 3×10^8 m s^{-1}, the same as the measured speed of light, showing that light is an electromagnetic wave.

In an isotropic, homogeneous, nonconducting, uncharged medium, where the permittivity and permeability are scalar quantities, Maxwell's equations can be written

$$\nabla \cdot \mathbf{E} = 0 \qquad 15.9.1$$

$$\nabla \cdot \mathbf{H} = 0 \qquad 15.9.2$$

$$\nabla \times \mathbf{H} = \varepsilon \dot{\mathbf{E}}. \qquad 15.9.3$$

$$\nabla \times \mathbf{E} = -\mu \dot{\mathbf{H}}. \qquad 15.9.4$$

Take the **curl** of equation 15.9.3, and make use of equation 15.6.4:

$$\mathbf{grad}\, \mathit{div}\ \mathbf{H} - \nabla^2 \mathbf{H} = \varepsilon \frac{\partial}{\partial t} \mathbf{curl}\, \mathbf{E}. \qquad 15.9.5$$

Substitute for *div* **H** and **curl E** from equations 15.9.2 and 15.9.4 to obtain

$$\nabla^2 \mathbf{H} = \varepsilon \mu \ddot{\mathbf{H}}. \qquad 15.9.6$$

Comparison with equation 15.1.2 shows that this is a wave of speed $1/\sqrt{\varepsilon\mu}$. (Verify that this has the dimensions of speed.)

In a similar manner the reader should easily be able to derive the equation

$$\nabla^2 \mathbf{E} = \varepsilon \mu \ddot{\mathbf{E}}. \qquad 15.9.7$$

In a vacuum, the speed is $1/\sqrt{\varepsilon_0 \mu_0}$. With $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} and $\varepsilon_0 = 8.854 \times 10^{-12}$ F m^{-1}, this comes to 2.998×10^8 m s^{-1}.

15.10 *Gauge Transformations*

We recall (equation 9.1.1) that a static electric field $\mathbf{E}$ can be derived from the negative of the gradient of a scalar potential function of space:

$$\mathbf{E} = -\mathbf{grad}\,V. \qquad 15.10.1$$

The zero of the potential is arbitrary. We can add any constant (with the dimensions of potential) to V. For example, if we define $V' = V + C$, where C is a constant (in the sense that it is not a function of x, y, z) then we can still calculate the electric field from $\mathbf{E} = -\mathbf{grad}\,V'$.

We also recall (equation 9.2.1) that a static magnetic field $\mathbf{B}$ can be derived from the **curl** of a magnetic vector potential function:

$$\mathbf{B} = \mathbf{curl}\,\mathbf{A}. \qquad 15.10.2$$

Let us also recall here the concept of the B-flux from equation 6.10.1:

$$\Phi_{\rm B} = \iint \mathbf{B}\cdot d\mathbf{A}. \qquad 15.10.3$$

It will be worth while here to recapitulate the dimensions and SI units of these quantities:

$\mathbf{E}$	$\mathrm{MLT^{-2}Q^{-1}}$	$\mathrm{V\ m^{-1}}$
$\mathbf{B}$	$\mathrm{MT^{-1}Q^{-1}}$	T
V	$\mathrm{ML^{2}T^{-2}Q^{-1}}$	V
$\mathbf{A}$	$\mathrm{MLT^{-1}Q^{-1}}$	T m or $\mathrm{Wb\ m^{-1}}$
$\Phi_{\rm B}$	$\mathrm{ML^{2}T^{-1}Q^{-1}}$	$\mathrm{T\ m^2}$ or Wb

Equation 15.10.2 is also true for a nonstatic field. Thus a time-varying magnetic field can be represented by the **curl** of a time-varying magnetic vector potential. However, we know from the phenomenon of electromagnetic induction that a varying magnetic field has the same effect as an electric field, so that, if the fields are not static, the electric field is the result of an electrical potential gradient and a varying magnetic field, so that equation 15.10.1 holds only for static fields.

If we combine the Maxwell equation $\mathbf{curl}\,\mathbf{E} = -\dot{\mathbf{B}}$ with the equation for the definition of the magnetic vector potential $\mathbf{curl}\ \mathbf{A} = \mathbf{B}$, we obtain $\mathbf{curl}(\mathbf{E} + \dot{\mathbf{A}}) = \mathbf{0}$. Then, since **curl grad** of any scalar function is zero, we can define a potential function V such that

$$\mathbf{E} + \dot{\mathbf{A}} = -\mathbf{grad}\,V. \qquad 15.10.4$$

(We could have chosen a plus sign, but we choose a minus sign so that it reduces to the familiar $\mathbf{E} = -\mathbf{grad}\,V$ for a static field.) Thus equations 15.10.4 and 15.10.2 define the electric and magnetic potentials – or at least they define the **grad**ient of V and the **curl** of **A**. But we recall that, in the static case, we can add an arbitrary constant to V (as long as the constant is dimensionally similar to V), and the equation $\mathbf{E} = -\mathbf{grad}\,V'$, where $V' = V + C$, still holds. Can we find a suitable transformation for V and **A** such that equations 15.10.2 and 15.10.4 still hold in the nonstatic case? Such a transformation would be a *gauge transformation.*

Let χ be some arbitrary scalar function of space and time. I demand little of the form of χ; indeed I demand only two things. One is that it is a "well-behaved" function, in the sense that it is everywhere and at all times single-valued, continuous and differentiable. The other is that it should have dimensions $\mathrm{ML^2T^{-1}Q^{-1}}$. This is the same as the dimensions of magnetic B-flux, but I am not sure that it is particularly helpful to think of this. It *will*, however, be useful to note that the dimensions of **grad** χ and of $\dot{\chi}$ are, respectively, the same as the dimensions of magnetic vector potential (**A**) and of electric potential (V).

Let us make the transformations

$$\mathbf{A'} = \mathbf{A} - \mathbf{grad}\,\chi \qquad 15.10.5$$

and

$$V' = V + \dot{\chi}. \qquad 15.10.6$$

We shall see very quickly that this transformation (and we have a wide choice in the form of χ) preserves the forms of equations 15.10.2 and 15.10.4, and therefore this transformation (or, rather, these transformations, since χ can have any well-behaved form) are *gauge transformations.*

Thus $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$ becomes $\mathbf{curl}(\mathbf{A'} + \mathbf{grad}\,\chi) = \mathbf{B}$. And since **curl grad** of any scalar field is zero, this becomes $\mathbf{curl}\,\mathbf{A'} = \mathbf{B}$.

Also, $\mathbf{grad}\,V = -(\mathbf{E} + \dot{\mathbf{A}})$

becomes $\mathbf{grad}(V' - \dot{\chi}) = -(\mathbf{E} + \dot{\mathbf{A}}' + \mathbf{grad}\dot{\chi})$, or $\mathbf{grad}\,V' = -(\mathbf{E} + \dot{\mathbf{A}}')$.

Thus the form of the equations is preserved. If we make a gauge transformation to the potentials such as equations 15.10.5 and 15.10.6, this does not change the fields **E** and **B**, so that the fields **E** and **B** are *gauge invariant.* Maxwell's equations in their usual form are expressed in terms of **E** and **B**, and are hence *gauge invariant*.

15.11 *Maxwell's Equations in Potential Form*

In their usual form, Maxwell's equations for an isotropic medium, written in terms of the fields, are

$$\mathrm{div}\,\mathbf{D} = \rho \tag{15.11.1}$$

$$\mathrm{div}\,\mathbf{B} = 0 \tag{15.11.2}$$

$$\mathbf{curl}\,\mathbf{H} = \dot{\mathbf{D}} + \mathbf{J} \tag{15.11.3}$$

$$\mathbf{curl}\,\mathbf{E} = -\dot{\mathbf{B}}. \tag{15.11.4}$$

If we write the fields in terms of the potentials:

$$\mathbf{E} = -\dot{\mathbf{A}} - \mathbf{grad}\,V \tag{15.11.5}$$

and

$$\mathbf{B} = \mathbf{curl}\,\mathbf{A}, \tag{15.11.6}$$

together with $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, we obtain for the first Maxwell equation, after some vector calculus and algebra,

★
$$\nabla^2 V + \frac{\partial}{\partial t}(\mathrm{div}\,\mathbf{A}) = -\frac{\rho}{\varepsilon}. \tag{15.11.7}$$

For the second equation, we merely verify that zero is equal to zero. (div **curl A** = 0.)

For the third equation, which requires a little more vector calculus and algebra, we obtain

★
$$\nabla^2\mathbf{A} - \varepsilon\mu\frac{\partial^2\mathbf{A}}{\partial t^2} = \mathbf{grad}\left(\mathrm{div}\,\mathbf{A} + \varepsilon\mu\frac{\partial V}{\partial t}\right) - \mu\mathbf{J}. \tag{15.11.8}$$

The speed of electromagnetic waves in the medium is $1/\sqrt{\varepsilon\mu}$, and, in a vacuum, equation 15.11.8 becomes

★
$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mathbf{grad}\left(\mathrm{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) - \mu_0\mathbf{J}, \tag{15.11.9}$$

where c is the speed of electromagnetic waves in a vacuum.

The fourth Maxwell equation, when written in terms of the potentials, tells us nothing new (try it), so equations 15.11.7 and 15.11.8 (or 15.11.9 *in vacuo*) are Maxwell's equations in potential form.

These equations look awfully difficult – but perhaps we can find a gauge transformation, using some form for χ, and subtracting **grad** ξ from **A** and adding $\dot{\xi}$ to V, which will make the equations much easier and which will still give the right answers for **E** and for **B**.

One of the things that make equations 15.11.7 and 15.11.9 look particularly difficult is that each equation contains both **A** and V; that is, we have two simultaneous differential equations to solve for the two potentials. It would be nice if we had one equation for **A** and one equation for V. This can be achieved, as we shall shortly see, if we can find a gauge transformation such that the potentials are related by

$$\operatorname{div}\mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}. \qquad 15.11.10$$

You should check that the two sides of this equation are dimensionally similar. What would be the SI units?

You'll see that this is chosen so as to make the "difficult" part of equation 15.11.9 zero.

If we make a gauge transformation and take the divergence of equation 15.10.5 and the time derivative of equation 15.10.6, we then see that condition 15.11.10 will be satisfied by a function χ that satisfies

$$\nabla^2\xi - \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} = -\operatorname{div}\mathbf{A'} - \frac{1}{c^2}\frac{\partial V'}{\partial t}. \qquad 15.11.11$$

Don't worry – you don't have to solve this equation and find the function χ; you just have to be assured that some such function exists such that, when applied to the potentials, the potentials will be related by equation 15.11.10. Then, if you substitute equation 15.11.10 into Maxwell's equations in potential form (equations 15.11.7 and 15.11.9), you obtain the following forms for Maxwell's equations *in vacuo* in potential form, and the **A** and V are now separated:

★ $$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \qquad 15.11.12$$

and ★ $$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \qquad 15.11.13$$

And, since these equations were arrived at by a gauge transformation, their solutions, when differentiated, will give the right answers for the fields.

15.12 *Retarded Potential*

In a static situation, in which the charge density ρ, the current density **J**, the electric field **E** and potential *V*, and the magnetic field **B** and potential **A** are all constant in time (i.e. they are functions of *x*, *y* and z but not of *t*) we already know how to calculate, *in vacuo*, the electric potential from the electric charge densityand the magnetic potential from the current density. The formulas are

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(x',y',z')\,dv'}{R} \qquad 15.12.1$$

and

$$\mathbf{A}(x,y,z) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(x',y',z')\,dv'}{R}\,. \qquad 15.12.2$$

Here *R* is the distance between the point (x',y',z') and the point $(x,\ y,\ z)$, and v' is a volume element at the point (x',y',z'). I can't remember if we have written these two equations in exactly that form before, but we have certainly used them, and given lots of examples of calculating *V* in Chapter 2, and one of calculating **A** in Section 9.3.

The question we are now going to address is whether these formulas are still valid in a nonstatic situation, in which the charge density ρ, the current density **J**, the electric field **E** and potential *V*, and the magnetic field **B** and potential **A** are all varying in time (i.e. they are functions of *x*, *y*, z and *t*). The answer is "yes, but…". The relevant formulas are indeed

$$V(x,y,z,t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(x',y',z',t')\,dv'}{R} \qquad 15.12.3$$

and

$$\mathbf{A}(x,y,z,t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(x',y',z',t')\,dv'}{R}\,, \qquad 15.12.4$$

…but notice the t' on the right hand side and the *t* on the left hand side! What this means is that, if $\rho(x',y',z',t')$ is the charge density at a point (x',y',z') at time t', equation 15.12.3 gives the correct potential at the point (x',y',z') at some *slightly later time t*, the time difference $t - t'$ being equal to the time *R*/*c* that it takes for an electromagnetic signal to travel from (x',y',z') to $(x,\ y,\ z)$. If the charge density at (x',y',z') changes, the information about this change cannot reach the point instantaneously; it takes a time *R*/*c* for the information to be transmitted from one point to another. The same considerations apply to the change in the magnetic potential when the current density changes, as described by equation 15.12.4. The potentials so calculated are called, naturally, the *retarded potentials.* While this result has been arrived at by a qualitative argument, in fact equations 15.12.3 and 4 can be obtained as a solution of the differential equations 15.11.12 and 13. Mathematically there is also a solution that gives an "advance potential" – that is, one in which $t' - t$ rather than $t - t'$ is equal to *R*/*c*. You can regard, if you wish, the retarded solution as the "physically acceptable" solution and discard the "advance" solution as not being physically significant. That is, the potential cannot predict in advance that the charge density is about to change, and so change its value before the charge density does. Alternatively one can think that the laws of physics, from the mathematical view at least, allow the universe to run equally well backward as well as forward, though in fact the arrow of time is such that cause must precede effect (a condition which, in relativity, leads to the conclusion that information cannot be transmitted from one place to another at a speed faster than the speed of light). One is also reminded that the laws of physics, from the mathematical view at least, allow the entropy of an isolated thermodynamical system to decrease (see Section 7.4 in the Thermodynamics part of these notes) – although in the real universe the arrow of time is such that the entropy in fact increases. Recall also the following passage from *Through the Looking-glass and What Alice Found There.*

Alice was just beginning to say, "There's a mistake somewhere -- -" when the Queen began screaming, so loud that she had to leave the sentence unfinished. "Oh, oh, oh!" shouted the Queen, shaking her hand about as if she wanted to shake it off. "My finger's bleeding! Oh, oh, oh, oh!"

Her screams were so exactly like the whistle of a steam-engine, that Alice had to hold both her hands over her ears.

"What *is* the matter?" she said, as soon as there was a chance of making herself heard. "Have you pricked your finger?"

"I haven't pricked it *yet*" the Queen said, "but I soon shall -- -oh, oh, oh!"

"When do you expect to do it?" Alice asked, feeling very much inclined to laugh.

"When I fasten my shawl again," the poor Queen groaned out: "the brooch will come undone directly. Oh, oh!" As she said the words the brooch flew open, and the Queen clutched wildly at it, and tried to clasp it again.

"Take care!" cried Alice. "You're holding it all crooked!" And she caught at the brooch; but it was too late: the pin had slipped, and the Queen had pricked her finger.

"That accounts for the bleeding, you see," she said to Alice with a smile. "Now you understand the way things happen here."

"But why don't you scream now?" Alice asked, holding her hands ready to put over her ears again. "Why, I've done all the screaming already," said the Queen. "What would be the good of having it all over again?"

Addendum. Coincidentally, just two days after having completed this chapter, I received the 2005 February issue of *Astronomy & Geophysics*, which included a fascinating article on the Arrow of Time. You might want to look it up. The reference is Davis, P., *Astronomy & Geophysics* (Royal Astronomical Society) **46**, 26 (2005).