CHAPTER 17
MAGNETIC DIPOLE MOMENT

17.1 *Introduction*

A number of different units for expressing magnetic dipole moment (hereafter simply "magnetic moment") are commonly seen in the literature, including, for example, erg G^{-1}, G cm^3, Oe cm^3, T m^3, A m^2, J T^{-1}. It is not always obvious how to convert from one to another, nor is it obvious whether all quantities described as "magnetic moment" refer to the same physical concept or are dimensionally or numerically similar. It can be almost an impossibility, for example, to write down a list of the magnetic moments of the planets in order of increasing magnetic moment if one refers to the diverse literature in which the moments of each of the nine planets are quoted in different units. This chapter explores some of these aspects of magnetic moment.

In previous chapters, I have used the symbols p_e and p_m for electric and magnetic dipole moment. In this chapter I shall be concerned exclusively with magnetic moment, and so I shall dispense with the subscript m.

17.2 *The SI Definition of Magnetic Moment*

If a magnet is placed in an external magnetic field **B**, it will experience a torque. The magnitude of the torque depends on the orientation of the magnet with respect to the magnetic field. There are two oppositely-directed orientations in which the magnet will experience the greatest torque, and the magnitude of the magnetic moment is **defined** as the *maximum torque experienced by the magnet when placed in unit external magnetic field.* The magnitude and direction of the torque is given by the equation

$$\tau = \mathbf{p}\times\mathbf{B}. \qquad 17.2.1$$

The SI unit for magnetic moment is clearly N m T^{-1}.

If an electric current I flows in a plane coil of area **A** (recall that area is a vector quantity – hence the boldface), the torque it will experience in a magnetic field is given by

$$\boldsymbol{\tau} = I\mathbf{A}\times\mathbf{B}. \qquad 17.2.2$$

This means that the magnetic moment of the coil is given by

$$\mathbf{p} = I\mathbf{A}. \qquad 17.2.3$$

Thus the unit A m^2 is also a correct SI unit for magnetic moment, though, unless the concept of "current in a coil" needs to be emphasized in a particular context, it is perhaps better to stick to N m T^{-1}.

While "J T^{-1}" is also formally dimensionally correct, it is perhaps better to restrict the unit "joule" to work or energy, and to use N m for torque. Although these are dimensionally similar, they are conceptually rather different. For this reason, the occasional practice seen in atomic physics of expressing magnetic moments in MeV T^{-1} is not entirely appropriate, however convenient it may sometimes seem to be in a field in which masses and momenta are often conveniently expressed in MeV/c^2 and MeV/c.

It is clear that the unit "T m^3", often seen for "magnetic moment" is not dimensionally correct for magnetic moment as defined above, so that, whatever quantity is being expressed by the often-seen "T m^3", it is not the conventionally defined concept of magnetic moment.

The *magnetization* **M** of a material is defined by the equation

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \qquad 17.2.4$$

Equations 17.2.2 and 17.2.4 for the definitions of magnetic moment and magnetization are consistent with the alternative concept of magnetization as "magnetic moment per unit volume".

17.3 *The Magnetic Field on the Equator of a Magnet*

By the "equator" of a magnet I mean a plane normal to its magnetic moment vector, passing through the mid-point of the magnet.

The magnetic field at a point at a distance r on the equator of a magnet may be expressed as a series of terms of successively higher powers of $1/r$ (the first term in the series being a term in r^{-3}), and the higher powers decrease rapidly with increasing distance. At large distances, the higher powers become negligible, so that, at a large distance from a small magnet, the magnitude of the magnetic field produced by the magnet is given approximately by

$$B = \frac{\mu_0}{4\pi}\frac{p}{r^3} . \qquad 17.3.1$$

For example, if the surface magnetic field on the equator of a planet has been measured, and the magnetic properties of the planet are being modelled in terms of a small magnet at the centre of the planet, the dipole moment can be calculated by multiplying the surface equatorial magnetic field by $\mu_0/(4\pi)$ times the cube of the radius of the planet. If B, μ_0 and r are expressed respectively in T, H m^{-1} and m, the magnetic moment will be in N m T^{-1}.

17.4 *CGS Magnetic Moment, and Lip Service to SI*

Equation 17.3.1 is the equation (written in the convention of quantity calculus, in which symbols stand for physical quantities rather than for their numerical values in some particular system of units) for the magnetic field at a large distance on the equator of a magnet. The equation is valid in <u>*any*</u> coherent system of units whatever, and its validity is not restricted to any particular system of units. Example of systems of units in which equation 17.3.1 are valid include SI, CGS EMU, and British Imperial Units.

If CGS EMU are used, the quantity $\mu_0/(4\pi)$ has the numerical value 1. Consequently, when working exclusively in CGS EMU, equation 17.3.1 is often written as

$$B = \frac{p}{r^3} . \qquad 17.4.1$$

This equation appears not to balance dimensionally. However, the equation is not written according to the conventions of quantity calculus, and the symbols do not stand for physical quantities. Rather, they stand for their numerical values in a particular system of units. Thus r is the distance in cm, B is the field in gauss, and p is the magnetic moment in dyne cm per gauss. However, because of the deceptive appearance of the equation, a common practice, for example, in calculating the magnetic moment of a planet is to measure its surface equatorial field, multiply it by the cube of the planet's radius, and then quote the magnetic moment in "G cm^3". While the numerical result is correct for the magnetic moment in CGS EMU, the units quoted are not.

While some may consider objections to incorrect units to be mere pedantry (and who would presumably therefore see nothing wrong with quoting a length in grams, as long as the actual number is correct), the situation becomes more difficult when a writer, wishing to pay lip service to SI, attempts to use equation 17.4.1 using SI units, by multiplying the surface equatorial field in T by the cube of the planet's radius, and then giving the magnetic moment in "T m^3", a clearly disastrous recipe!

Of course, some may use equation 17.4.1 as a *definition* of magnetic moment. If that is so, then the quantity so defined is clearly not the same quantity, physically, conceptually, dimensionally or numerically, as the quantity defined as magnetic moment in Section 17.2.

17.5 *Possible Alternative Definitions of Magnetic Moment*

Although the standard SI definition of magnetic moment is described in Section 17.2, and there is little reason for anyone who wishes to be understood by others to use any other, the previous paragraph suggested that there might be more than one choice as to how one wishes to define magnetic moment. Do we use equation 17.2.1 or equation 17.4.1 as the definition? (They are clearly different concepts.) Additional degrees of freedom as to how one might choose to define magnetic moment depend on whether we elect to use magnetic field H or magnetic field B in the definition, or whether the permeability is or is not to include the factor 4π in its definition – that is, whether we elect to use a "rationalized" or "unrationalized" definition of permeability.

If one chooses to define the magnetic moment as the maximum torque experienced in unit external magnetic field, there is still a choice as to whether by magnetic field we choose H or B. Thus we could define magnetic moment by either of the following two equations:

$$\tau = p_1 H \qquad 17.5.1$$

or

$$\tau = p_2 B. \qquad 17.5.2$$

Alternatively, we could choose to define the magnetic moment is terms of the field on the equator. In that case we have a choice of four. We can choose to use B or H for the magnetic field, and we can choose to exclude or include 4π in the denominator:

$$B = \frac{p_3}{r^3}, \qquad 17.5.3$$

$$H = \frac{p_4}{r^3}, \qquad 17.5.4$$

$$B = \frac{p_5}{4\pi r^3}, \qquad 17.5.5$$

$$H = \frac{p_6}{4\pi r^3}. \qquad 17.5.6$$

These six possible definitions of magnetic moment are clearly different quantities, and one may well wonder why to list them all. The reason is that *all* of them are to be found in current scientific literature. To give some hint as to the unnecessary complications introduced when authors depart from the simple SI definition, I list in Table XVII.1 the *dimensions* of each version of magnetic moment, the CGS EM unit, the SI unit, and the conversion factor between CGS and SI. The conversion factors cannot be obtained simply by referring to the *dimensions*, because this does not take into account the inclusion or exclusion of 4π in the permeability. The correct factors can be obtained from the *units*, for example by noting that 1 Oe = $10^{-3}/(4\pi)$ A m^{-1} and 1 G = 10^{-4} T.

TABLE XVII.1
DIMENSIONS, CGS AND SI UNITS, AND CONVERSION FACTORS
FOR MAGNETIC MOMENTS

	Dimensions	1 CGS EMU	=	Conversion factor	SI unit
p_1	$ML^3T^{-1}Q^{-1}$	1 dyn cm Oe^{-1}	=	$4\pi \times 10^{-10}$	N m $(A/m)^{-1}$
p_2	$L^2T^{-1}Q$	1 dyn cm G^{-1}	=	10^{-3}	N m T^{-1}
p_3	$ML^3T^{-1}Q^{-1}$	1 G cm^3	=	10^{-10}	T m^3
p_4	$L^2T^{-1}Q$	1 Oe cm^3	=	$10^{-3}/4\pi$	A m^2
p_5	$ML^3T^{-1}Q^{-1}$	1 G cm^3	=	10^{-10}	T m^3
p_6	$L^2T^{-1}Q$	1 Oe cm^3	=	$10^{-3}/4\pi$	A m^2

17.6 *Thirteen Questions*

We have seen that the SI definition of magnetic moment is unequivocally defined as the maximum torque experienced in unit external field. Nevertheless some authors prefer to think of magnetic moment as the product of the equatorial field and the cube of the distance. Thus there are two conceptually different concepts of magnetic moment, and, when to these are added minor details as to whether the magnetic field is B or H, and whether or not the permeability should include the factor 4π, six possible definitions of magnetic moment, described in Section 17.6, all of which are to be found in current literature, arise.

Regardless, however, how one chooses to define magnetic moment, whether the SI definition or some other unconventional definition, it should be easily possible to answer both of the following questions:

A. Given the magnitude of the equatorial field on the equator of a magnet, what is the maximum torque that that magnet would experience if it were placed in an external field?

B. Given the maximum torque that a magnet experiences when placed in an external field, what is the magnitude of the equatorial field produced by the magnet?

It must surely be conceded that a failure to be able to answer such basic questions indicates a failure to understand what is meant by magnetic moment.

I therefore now ask a series of thirteen questions. The first six are questions of type A, in which I use the six possible definitions of magnetic moment. The next six are similar questions of type B. And the last is an absurdly simple question, which anyone who believes he understands the meaning of magnetic moment should easily be able to answer.

1. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 cm is 1 Oe.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 Oe, and what is its magnetic moment?

Note that, in this question and the following seven there *must* be a unique answer for the *torque.* The answer you give for the *magnetic moment*, however, will depend on how you choose to define magnetic moment, and on whether you choose to give the answer in SI units or CGS EMU.

2. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 cm is 1 Oe.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 G, and what is its magnetic moment?

3. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 cm is 1 G.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 Oe, and what is its magnetic moment?

4. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 cm is 1 G.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 G, and what is its magnetic moment?

5. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 m is 1 A m^{-1}.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 A m^{-1}, and what is its magnetic moment?

6. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 m is 1 A m^{-1}.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 T, and what is its magnetic moment?

7. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 m is 1 T.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 A m^{-1}, and what is its magnetic moment?

8. The magnitude of the field in the equatorial plane of a magnet at a distance of 1 m is 1 T.

What is the maximum torque that this magnet will experience in an external magnetic field of 1 T, and what is its magnetic moment?

9. A magnet experiences a maximum torque of 1 dyn cm if placed in a field of 1 Oe. What is the magnitude of the field in the equatorial plane at a distance of 1 cm, and what is the magnetic moment?

Note that, in this question and the following three there *must* be a unique answer for B and a unique answer for H, though each can be expressed in SI or in CGS EMU, while the answer for the magnetic moment depends on which definition you adopt.

10. A magnet experiences a maximum torque of 1 dyn cm if placed in a field of 1 G. What is the magnitude of the field in the equatorial plane at a distance of 1 cm, and what is the magnetic moment?

11. A magnet experiences a maximum torque of 1 N m if placed in a field of 1 A m^{-1}. What is the magnitude of the field in the equatorial plane at a distance of 1 m, and what is the magnetic moment?

12. A magnet experiences a maximum torque of 1 N m if placed in a field of 1 T. What is the magnitude of the field in the equatorial plane at a distance of 1 m, and what is the magnetic moment?

I'll pose Question Number 13 a little later. In the meantime the answers to the first four questions are given in Table XVII.2, and the answers to Questions 5 – 12 are given in

Tables XVII.3 and 4. The sheer complexity of these answers to absurdly simple questions is a consequence of different usages by various authors of the meaning of "magnetic moment" and of departure from standard SI usage.

TABLE XVII.2
ANSWERS TO QUESTIONS 1 – 4 IN CGS EMU AND SI UNITS
The answers to the first four questions are identical

τ	=	1 dyn cm	=	10^{-7} N m
p_1	=	1 dyn cm Oe^{-1}	=	$4\pi \times 10^{-7}$ N m $(\text{A/m})^{-1}$
p_2	=	1 dyn cm G^{-1}	=	10^{-3} N m $(\text{T})^{-1}$
p_3	=	1 G cm^3	=	10^{-10} T m^3
p_4	=	1 Oe cm^3	=	$10^{-3}/(4\pi)$ A m^2
p_5	=	4π G cm^3	=	$4\pi \times 10^{-10}$ T m^3
p_6	=	4π Oe cm^3	=	10^{-3} A m^2

TABLE XVII.3
ANSWERS TO QUESTIONS 5 – 8 IN CGS EMU AND SI UNITS

		5	6	7	8	
τ	=	$(4\pi)^2$	$4\pi \times 10^7$	$4\pi \times 10^7$	10^{14}	dyn cm
	=	$(4\pi)^2 \times 10^{-7}$	4π	4π	10^7	N m
p_1	=	$4\pi \times 10^3$	$4\pi \times 10^3$	10^{10}	10^{10}	dyn cm Oe^{-1}
	=	$(4\pi)^2 \times 10^{-7}$	$(4\pi)^2 \times 10^{-7}$	4π	4π	N m $(\mathrm{A/m})^{-1}$
p_2	=	$4\pi \times 10^3$	$4\pi \times 10^3$	10^{10}	10^{10}	dyn cm G^{-1}
	=	4π	4π	10^7	10^7	N m T^{-1}
p_3	=	$4\pi \times 10^3$	$4\pi \times 10^3$	10^{10}	10^{10}	G cm^3
	=	$4\pi \times 10^{-7}$	$4\pi \times 10^{-7}$	1	1	T m^3
p_4	=	$4\pi \times 10^3$	$4\pi \times 10^3$	10^{10}	10^{10}	Oe cm^3
	=	1	1	$10^7/(4\pi)$	$10^7/(4\pi)$	A m^2
p_5	=	$(4\pi)^2 \times 10^3$	$(4\pi)^2 \times 10^3$	$4\pi \times 10^{10}$	$4\pi \times 10^{10}$	G cm^3
	=	$(4\pi)^2 \times 10^{-7}$	$(4\pi)^2 \times 10^{-7}$	4π	4π	T m^3
p_6	=	$(4\pi)^2 \times 10^3$	$(4\pi)^2 \times 10^3$	$4\pi \times 10^{10}$	$4\pi \times 10^{10}$	Oe cm^3
	=	4π	4π	10^7	10^7	A m^2

TABLE XVII.4
ANSWERS TO QUESTIONS 9 – 12 IN CGS EMU AND SI UNITS

		9	10	11	12	
B	=	1	1	$10^4/(4\pi)$	10^{-3}	G
	=	10^{-4}	10^{-4}	$1/(4\pi)$	10^{-7}	T
H	=	1	1	$10^4/(4\pi)$	10^{-3}	Oe
	=	$10^3/(4\pi)$	$10^3/(4\pi)$	$10^7/(4\pi)^2$	$1/(4\pi)$	A m^{-1}
p_1	=	1	1	$10^{10}/(4\pi)$	10^3	dyn cm Oe^{-1}
	=	$4\pi \times 10^{-10}$	$4\pi \times 10^{-10}$	1	$4\pi\times10^{-7}$	N m $(A/m)^{-1}$
p_2	=	1	1	$10^{10}/(4\pi)$	10^3	dyn cm G^{-1}
	=	10^{-3}	10^{-3}	$10^7/(4\pi)$	1	N m T^{-1}
p_3	=	1	1	$10^4/(4\pi)$	10^{-3}	G cm^3
	=	10^{-10}	10^{-10}	$10^{-6}/(4\pi)$	10^{-13}	T m^3
p_4	=	1	1	$10^4/(4\pi)$	10^{-3}	Oe cm^3
	=	$10^{-3}/(4\pi)$	$10^{-3}/(4\pi)$	$10/(4\pi)^2$	$10^{-6}/(4\pi)$	A m^2
p_5	=	4π	4π	10^4	$4\pi \times 10^{-3}$	G cm^3
	=	$4\pi \times 10^{-10}$	$4\pi \times 10^{-10}$	10^{-6}	$4\pi \times 10^{-13}$	T m^3
p_6	=	4π	4π	10^4	$4\pi \times 10^{-3}$	Oe cm^3
	=	10^{-3}	10^{-3}	$10/(4\pi)$	10^{-6}	A m^2

The thirteenth and last of these questions is as follows: Assume that Earth is a sphere of radius 6.4×10^6 m = 6.4×10^8 cm, and that the surface field at the magnetic equator is $B = 3 \times 10^{-5}$ T = 0.3 G, or $H = 75/\pi \text{ A m}^{-1}$ = 0.3 Oe, what is the magnetic moment of Earth? It is hard to imagine a more straightforward question, yet it would be hard to find two people who would give the same answer.

The SI answer (which, to me, is the only answer) is

$$B = \frac{\mu_0}{4\pi}\frac{p}{r^3}, \quad \therefore \; p = \frac{4\pi r^3 B}{\mu_0} = \frac{4\pi \times (6.4\times10^6)^3 \times 3\times10^{-5}}{4\pi\times10^{-7}} = 7.86 \times 10^{22} \text{ N m T}^{-1}.$$

This result correctly predicts that, if Earth were placed in an external field of 1 T, it would experience a maximum torque of 7.86×10^{22} N m, and this is the normal meaning of what is meant by magnetic moment.

A calculation in GCS might proceed thus:

$$B = \frac{p}{r^3}, \quad \therefore \; p = r^3 B = (6.4\times10^8)^3 \times 0.3 = 7.86 \times 10^{25} \text{ G cm}^3.$$

Is this the same result as was obtained from the SI calculation? We can use the conversions 1 G = 10^{-4} T and 1 cm^3 = 10^{-6} m^3, and we obtain

$$p = 7.86 \times 10^{15} \text{ T m}^3.$$

We arrive at a number that not only differs from the SI calculation by 10^7, but is expressed in quite different, dimensionally dissimilar, units.

Perhaps the CGS calculation should be

$$H = \frac{p}{r^3}, \quad \therefore \; p = r^3 H = (6.4\times10^8)^3 \times 0.3 = 7.86 \times 10^{25} \text{ Oe cm}^3.$$

Now 1 Oe = $1000/(4\pi)$ A m^{-1} and 1 cm^3 = 10^{-6} m^3, and we obtain

$$p = 6.26 \times 10^{21} \text{ A m}^2$$

This time we arrive at SI units that are dimensionally similar to N m T^{-1}, and which are perfectly correct SI units, but the magnetic moment is smaller than correctly predicted by the SI calculation by a factor of 12.6.

Yet again, we might do what appears to be frequently done by planetary scientists, and we can multiply the surface field in T by the cube of the radius in m to obtain

$$p = 7.86 \times 10^{15} \quad \text{T m}^3.$$

This arrives at the same result as one of the CGS calculations, but, whatever it is, it is not the magnetic moment in the sense of the greatest torque in a unit field. The quantity so obtained appears to be nothing more that the product of the surface equatorial field and the cube of the radius, and as such would appear to be a purposeless and meaningless calculation.

It would be a good deal more meaningful merely to multiply the surface value of H by 3. This in fact would give (correctly) the dipole moment divided by the volume of Earth, and hence it would be the average *magnetization* of Earth – a very meaningful quantity, which would be useful in comparing the magnetic properties of Earth with those of the other planets.

17.7 *Additional Remarks*

The units erg G^{-1} or J T^{-1} are frequently encountered for magnetic moment. These may be dimensionally correct, although ergs and joules (units of energy) are not quite the same things as dyn cm or N m as units of torque. It could be argued that magnetic moment could be defined from the expression $-\boldsymbol{p}.\boldsymbol{B}$ for the potential energy of a magnet in a magnetic field. But the correct expression is actually constant $-\boldsymbol{p}.\boldsymbol{B}$, the constant being zero only if you specify that the energy is taken to be zero when the magnetic moment and field vectors are perpendicular to each other. This seems merely to add yet further complications to what should be, but unfortunately is not, a concept of the utmost simplicity.

Nevertheless the use of ergs or joules rather than dyn cm or N m is not uncommon, and nuclear and particle physicists commonly convert joules to MeV. Magnetic moments of atomic nuclei are commonly quoted in nuclear magnetons, where a nuclear magneton is $e\hbar/(2m_p)$ and has the value 3.15×10^{-4} MeV T^{-1}. While one is never likely to want to express the magnetic moment of the planet Uranus in nuclear magetons, it is sobering to attempt to do so, given that the magnetic moment of Uranus is quoted as 0.42 Oe km^{-1}. While on the subject of Uranus, I have seen it stated that the magnetic quadrupole of Uranus is or the same order of magnitude as its magnetic dipole moment – though, since these are dimensionally dissimilar quantities, such a statement conveys no meaning.

Another exercise to illustrate the points I have been trying to make is as follows. From four published papers I find the following. The magnetic moment of Mercury is 1.2×10^{19} A m^2 in one paper, and 300 nT R_M^3 in another. The magnetic moment of Uranus is 4.2×10^{12} Oe km^3 in one paper, and 0.23 G R_U^3 in another. The radii of Mercury and Uranus are, respectively, 2.49×10^6 m and 2.63×10^7 m. Calculate the ratio of the magnetic moment of Uranus to that of Mercury. If you are by now completely confused, you are not alone.

17.8 *Conclusion*

Readers will by now probably be bewildered at the complexities described in this chapter. After all, there could scarcely be a simpler notion than that of the torque experienced by a magnet in a magnetic field, and there would seem to be no need for all of these complicated variations. You are right – there is no such need. All that need be known is summarized in Sections 17.2 and 17.3. The difficulty arises because authors of scientific papers are using almost all possible variations of what they think is meant by magnetic moment, and this has led to a thoroughly chaotic situation. All I can do is to hope that readers of these notes will be encouraged to use only the standard SI definition and units for magnetic moment, and to be aware of the enormous complications arising when they depart from these.