CHAPTER 1
REFLECTION AND REFRACTION

1.1 *Introduction*

This "book" is not intended to be a vast, definitive treatment of everything that is known about geometric optics. It covers, rather, the geometric optics of first-year students, whom it will either help or confuse yet further, though I hope the former. The part of geometric optics that often causes the most difficulty, particularly in getting the right answer for homework or examination problems, is the vexing matter of sign conventions in lens and mirror calculations. It seems that no matter how hard we try, we always get the sign wrong! This aspect will be dealt with in Chapter 2. The present chapter deals with simpler matters, namely reflection and refraction at a plane surface, except for a brief foray into the geometry of the rainbow. The rainbow, of course, involves refraction by a spherical drop. For the calculation of the radius of the bow, only Snell's law is needed, but some knowledge of physical optics will be needed for a fuller understanding of some of the material in section 1.7, which is a little more demanding than the rest of the chapter.

1.2 *Reflection at a Plane Surface*

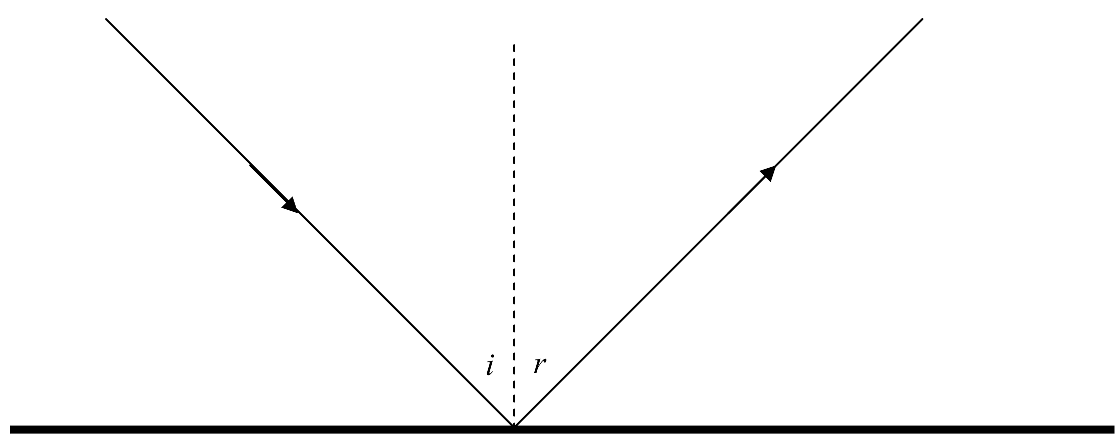


FIGURE I.1

The law of reflection of light is merely that the angle of reflection r is equal to the angle of incidence r. There is really very little that can be said about this, but I'll try and say what little need be said.

i. It is customary to measure the angles of incidence and reflection from the normal to the reflecting surface rather than from the surface itself.

ii. Some curmudgeonly professors may ask for the law*S* of reflection, and will give you only half marks if you neglect to add that the incident ray, the reflected ray and the normal are coplanar.

iii. A plane mirror forms a *virtual image* of a *real object*:

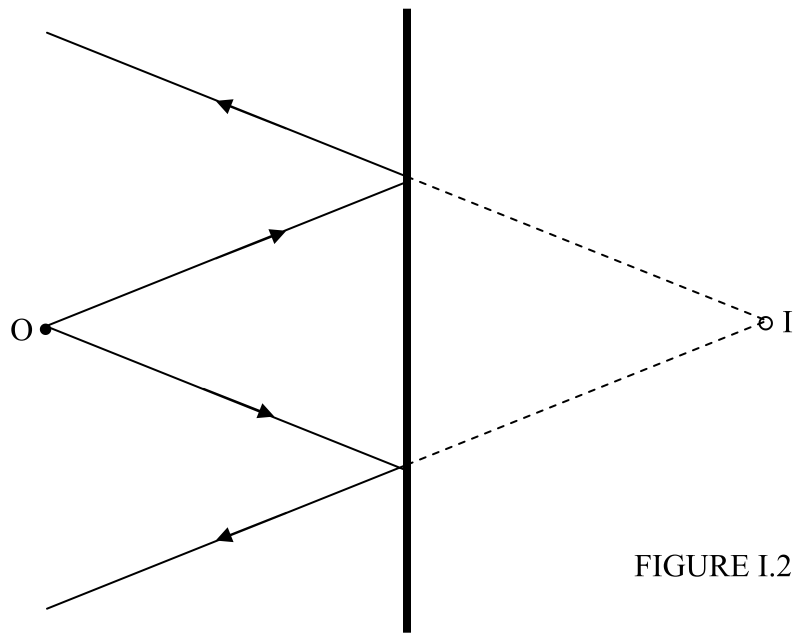


FIGURE I.2

or a *real image* of a *virtual object*:

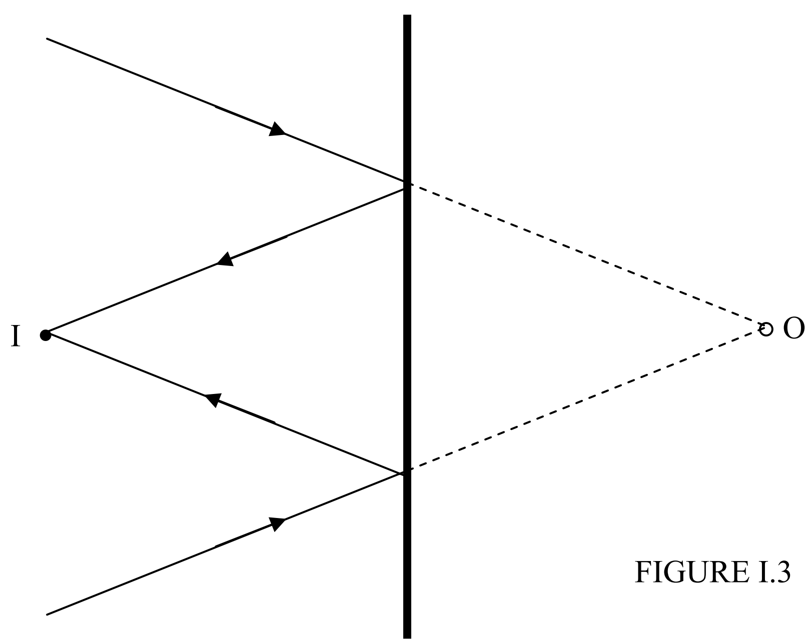


FIGURE I.3

iv. It is usually said that the image is as far behind the mirror as the object is in front of it. In the case of a virtual object (i.e. light converging on the mirror, presumably from some large lens somewhere to the left) you'd have to say that the image is as far *in front of* the mirror as the object is *behind it*!

v. If the mirror were to move at speed v away from a real object, the virtual image would move at speed $2v$. I'll leave you to think about what happens in the case of a virtual object.

vi. If the mirror were to rotate through an angle θ (or were to rotate at an angular speed ω), the reflected ray would rotate through an angle 2θ (or at an angular speed 2ω).

vii. Only smooth, shiny surfaces reflect light as described above. Most surfaces, such as paper, have minute irregularities on them, which results in light being scattered in many directions. Various equations have been proposed to describe this sort of scattering. If the reflecting surface looks equally bright when viewed from all directions, the surface is said to be a perfectly diffusing Lambert's law surface. Reflection according to the $r = i$ law of reflection, with the incident ray, the reflected ray and the normal being coplanar, is called *specular* reflection (Latin: speculum, a mirror). Most surfaces are intermediate between specular reflectors and perfectly diffusing surfaces. This chapter deals exclusively with specular reflection.

viii. The image in a mirror is reversed from left to right, and from back to front, but is not reversed up and down. Discuss.

ix. If you haven't read *Through the Looking-glass and What Alice Found There*, you are missing something.

1.3 *Refraction at a Plane Surface*

I was taught Snell's Law of Refraction thus:

When a ray of light enters a denser medium it is refracted towards the normal in such a manner than the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant, this constant being called the *refractive index n*.

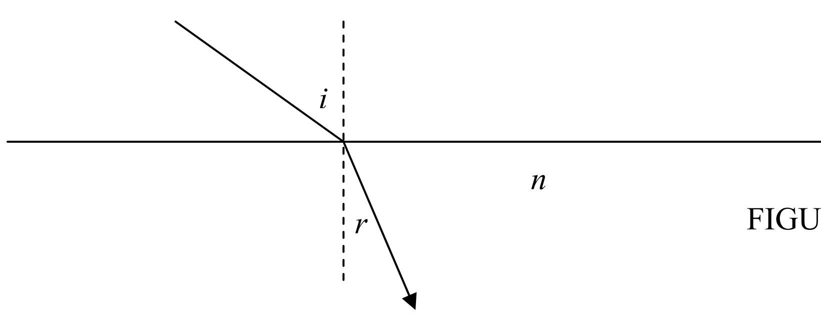


FIGURE I.4

This is all right as far as it goes, but we may be able to do better.

i. Remember the curmudgeonly professor who will give you only half marks unless you also say that the incident ray, the refracted ray and the normal are coplanar.

ii. The equation

$$\frac{\sin i}{\sin r} = n, \qquad 1.3.1$$

where n is the refractive index of the medium, is all right as long as the light enters the medium from a vacuum. The refractive index of air is very little different from unity. Details on the refractive index of air may be found in my notes on Stellar Atmospheres (chapter 7, section 7.1) and Celestial Mechanics (subsection11.3.3).

If light is moving from one medium to another, the law of refraction takes the form

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 . \qquad 1.3.2$$

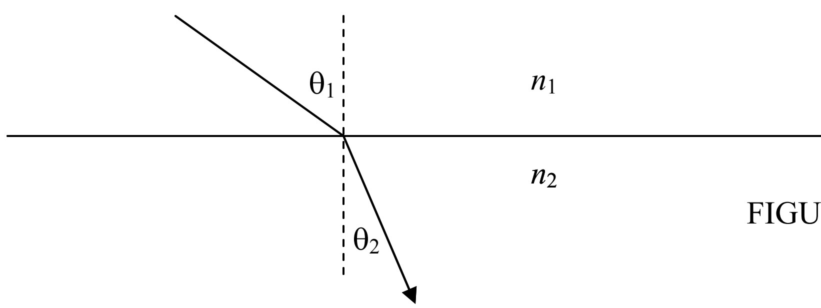


FIGURE I.5

iii. The statement of Snell's law as given above implies, if taken literally, that there is a one-to-one relation between refractive index and density. There must be a formula relating refractive index and density. If I tell you the density, you should be able to tell me the refractive index. And if I tell you the refractive index, you should be able to tell me the density. If you arrange substances in order of increasing density, this will also be their order of increasing refractive index.

This is not quite true, and, if you spend a little while looking up densities and refractive indices of substances in, for example, the *CRC Handbook of Physics and Chemistry*, you will find many examples of less dense substances having a higher refractive index than more dense substances. It is true in a general sense usually that denser substances have higher indices, but there is no one-to-one correspondence.

In fact light is bent towards the normal in a "denser" medium as a result of its slower speed in that medium, and indeed the speed υ of light in a medium of refractive index n is given by

$$n = c/\upsilon , \qquad 1.3.3$$

where c is the speed of light *in vacuo*. Now the speed of light in a medium is a function of the electrical permittivity ε and the magnetic permeability μ:

$$\upsilon = 1/\sqrt{\varepsilon\mu}\ . \qquad 1.3.4$$

The permeability of most nonferromagnetic media is very little different from that of a vacuum, so the refractive index of a medium is given approximately by

$$n \approx \sqrt{\frac{\varepsilon}{\varepsilon_0}}\ . \qquad 1.3.5$$

Thus there is a much closer correlation between refractive index and relative permittivity (dielectric constant) than between refractive index and density. Note, however, that this is only an approximate relation. In the detailed theory there is a small dependence of the speed of light and hence refractive index on the frequency (hence wavelength) of the light. Thus the refractive index is greater for violet light than for red light (*violet* light is refracted more *violently*). The splitting up of white light into its constituent colours by refraction is called *dispersion.*

1.4 *Real and Apparent Depth*

When we look down into a pool of water from above, the pool looks less deep than it really is. Figure I.6 shows the formation of a virtual image of a point on the bottom of the pool by refraction at the surface.

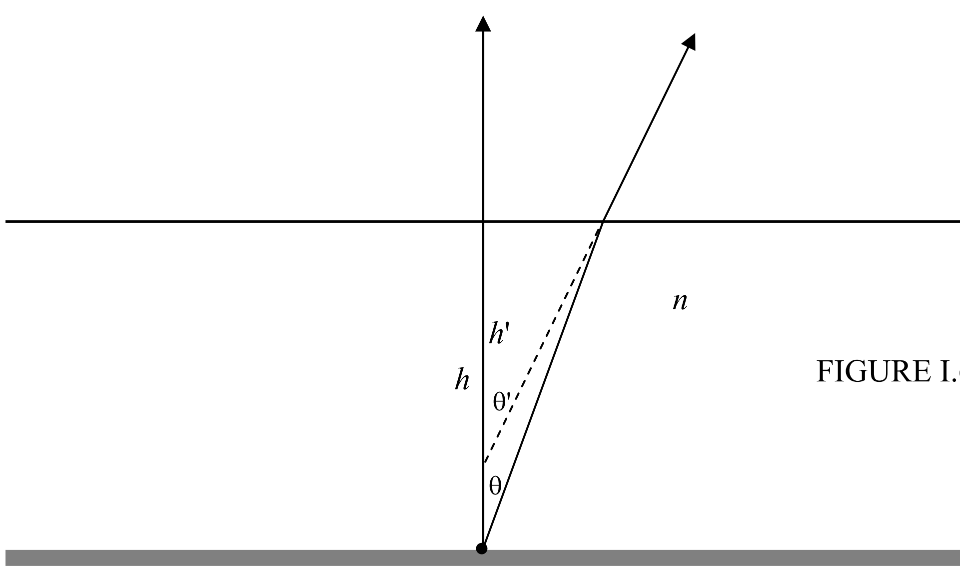


FIGURE I.6

The diameter of the pupil of the human eye is in the range 4 to 7 mm, so, when we are looking down into a pool (or indeed looking at anything that is not very close to our eyes), the angles involved are small. Thus in figure I.6 you are asked to imagine that all the angles are small; actually to draw them small would make for a very cramped drawing. Since angles are small, I can approximate Snell's law by

$$n \approx \frac{\tan\theta'}{\tan\theta} \qquad 1.4.1$$

and hence

$$\frac{\text{real depth}}{\text{apparent depth}} = \frac{h}{h'} = \frac{\tan\theta'}{\tan\theta} = n. \qquad 1.4.2$$

For water, n is about $4/3$, so that the apparent depth is about ¾ of the real depth.

Exercise. An astronomer places a photographic film, or a CCD, at the primary focus of a telescope. He then decides to insert a glass filter, of refractive index n and thickness t, in front of the film (or CCD). In which direction should he move the film or CCD, and by how much, so that the image remains in focus?

Now if Snell's law really were given by equation 1.4.1, all refracted rays from the object would, when produced backwards, appear to diverge from a single point, namely the virtual image. But Snell's law is really $n = \frac{\sin\theta'}{\sin\theta}$, so what happens if we do not make the small angle approximation?

We have $\frac{h}{h'} = \frac{\tan\theta'}{\tan\theta}$ and, if we apply the trigonometric identity $\tan\theta = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ and apply Snell's law, we find that

$$\frac{h}{h'} = \frac{n\cos\theta}{\sqrt{1-n^2\sin^2\theta}}. \qquad 1.4.3$$

Exercise. Show that, to first order in θ this becomes $h/h' = n$.

Equation 1.4.3 shows h' is a function of θ – that the refracted rays, when projected backwards, do not all appear to come from a single point. In other words, a point object does not result in a point image. Figure I.7 shows (for n = 1.5 – i.e. glass rather than water) the backward projections of the refracted rays for θ' = 15, 30, 45, 60 and 75 degrees, together with their envelope or "caustic curve". The "object" is at the bottom left corner of the frame, and the surface is the upper side of the frame.

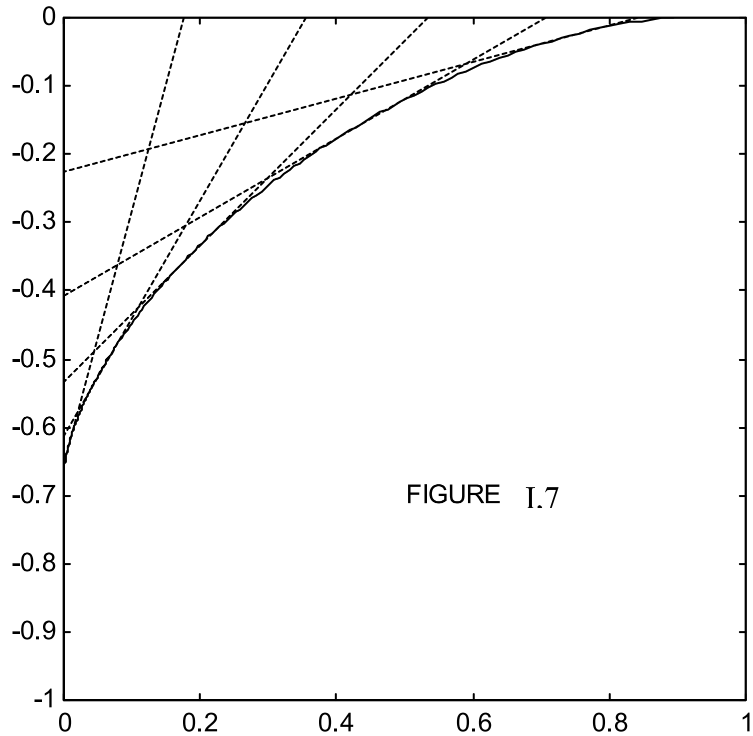


FIGURE I.7

Exercise (for the mathematically comfortable). Show that the parametric equations for the caustic curve are

$$x - y\tan\theta' - h\tan\theta = 0 \qquad 1.4.4$$

and

$$ny\sec^3\theta' + h\sec^2\theta = 0. \qquad 1.4.5$$

Here, $y = 0$ is taken to be the refractive surface, and θ and θ' are related by Snell's law.

Thus refraction at a plane interface produces an *aberration* in the sense that light from a point object does not diverge from a point image. This type of aberration is somewhat similar to the type of aberration produced by reflection from a spherical mirror, and to that extent the aberration could be referred to as "spherical aberration". If a point at the bottom of a pond is viewed at an angle to the surface, rather than perpendicular to it, a further aberration called "astigmatism" is produced. If I write a chapter on aberrations, this will be included there.

1.5 *Reflection and Refraction*

We have described reflection and refraction, but of course when a ray of light encounters an interface between two transparent media, a portion of it is reflected and a portion is refracted, and it is natural to ask, even during an early introduction to the subject, just what fraction is reflected and what fraction is refracted. The answer to this is quite complicated, for it takes depends not only on the angle of incidence and on the two refractive indices, but also on the initial state of polarization of the incident light; it takes us quite far into electromagnetic theory and is beyond the scope of this chapter, which is intended to deal largely with just the *geometry* of reflection and refraction. However, since it is a natural question to ask, I can give explicit formulas for the fractions that are reflected and refracted in the case where the incident light in unpolarized.

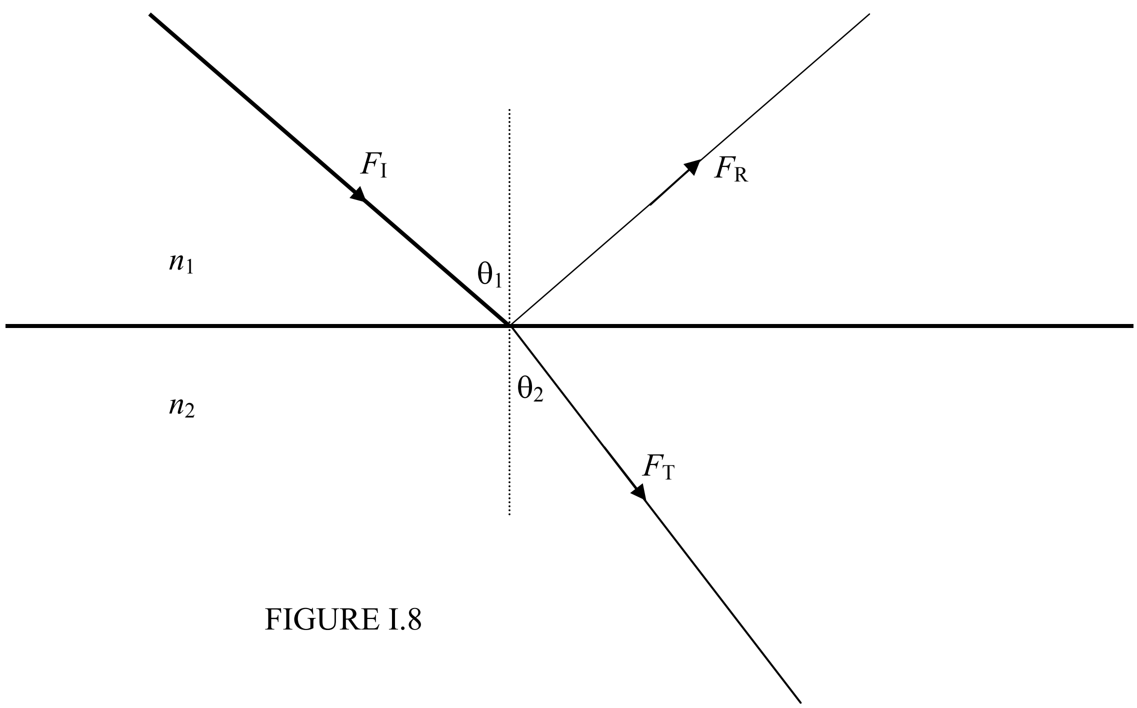


FIGURE I.8

Figure I.8 shows an incident ray of energy flux density (W m^{-2} normal to the direction of propagation) $F_{\rm I}$ arriving at an interface between media of indices n_1 and n_2. It is subsequently divided into a reflected ray of flux density $F_{\rm R}$ and a transmitted ray of flux density $F_{\rm T}$. The fractions transmitted and reflected (t and r) are

$$t = \frac{F_{\rm T}}{F_{\rm I}} = 2n_1n_2\cos\theta_1\cos\theta_2\left(\frac{1}{(n_1\cos\theta_1+n_2\cos\theta_2)^2} + \frac{1}{(n_1\cos\theta_2+n_2\cos\theta_1)^2}\right) \quad 1.5.1$$

and

$$r = \frac{F_{\mathrm{R}}}{F_{\mathrm{I}}} = \frac{1}{2}\left[\left(\frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}\right)^2 + \left(\frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}\right)^2\right]. \qquad 1.5.2$$

Here the angles and indices are related through Snell's law, equation 1.3.2. If you have the energy, show that the sum of these is 1.

Both the transmitted and the reflected rays are partially plane polarized. If the angle of incidence and the refractive index are such that the transmitted and reflected rays are perpendicular to each other, the reflected ray is completely plane polarized – but such details need not trouble us in this chapter.

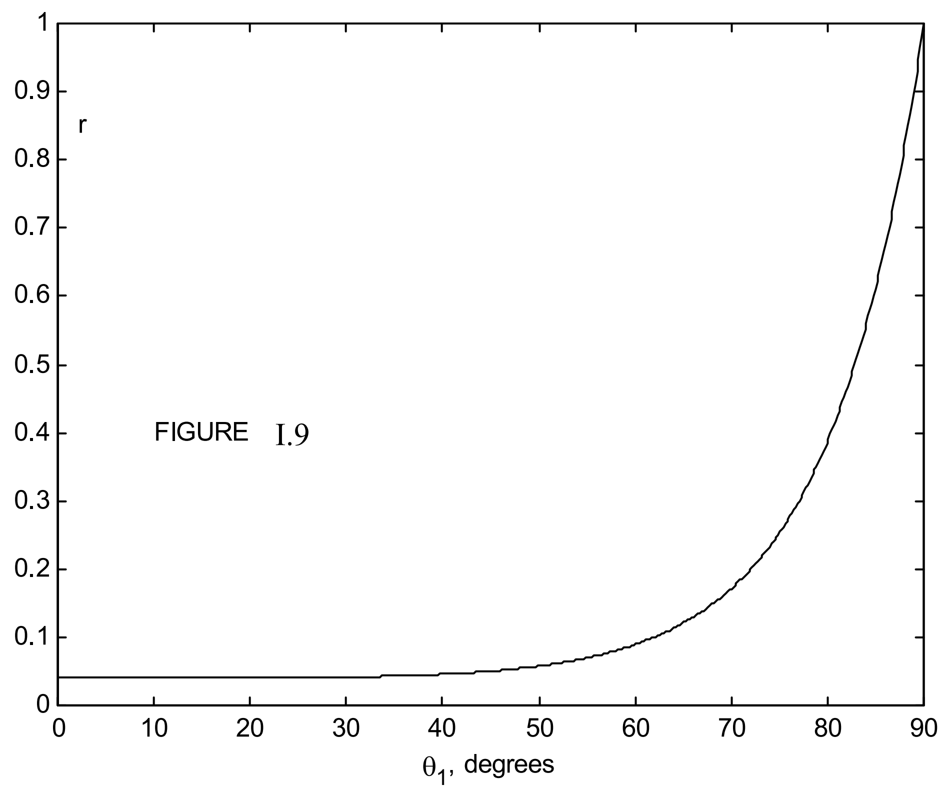
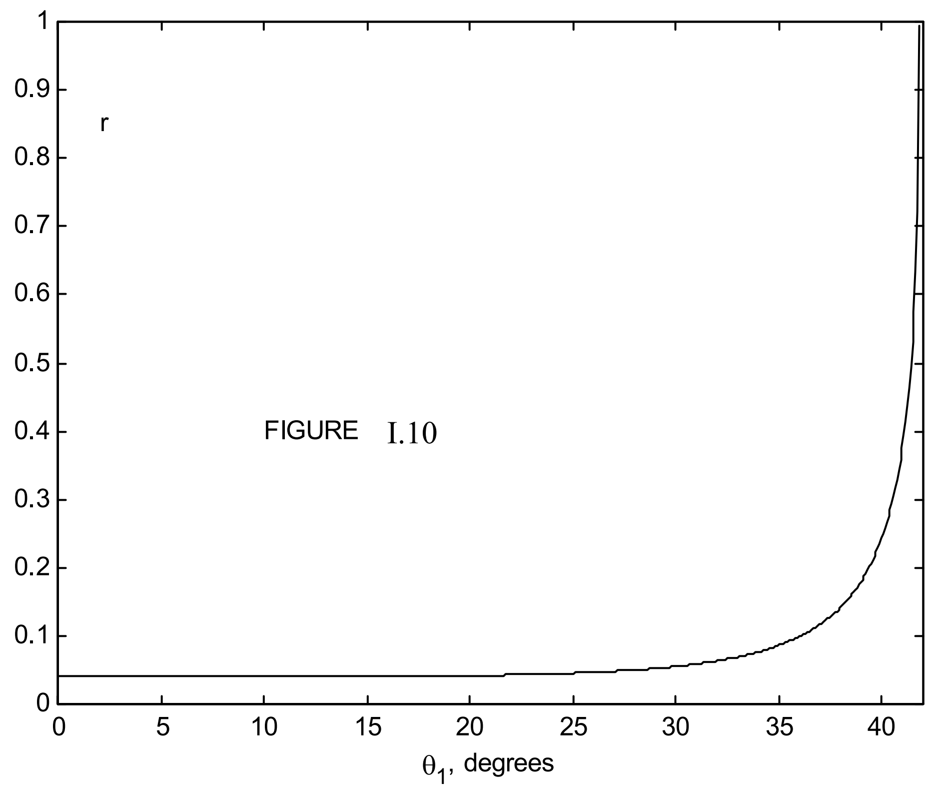


Figure I.9 shows the reflection coefficient as a function of angle of incidence for unpolarized incident light with $n_1 = 1.0$ and $n_2 = 1.5$ (e.g. glass). Since $n_2 > n_1$, we have *external reflection.* We see that for angles of incidence less than about 45 degrees, very little of the light is reflected, but after this the reflection coefficient increases rapidly with angle of incidence, approaching unity as $\theta_1 \rightarrow 90^{\circ}$ (grazing incidence).

If $n_1 = 1.5$ and $n_2 = 1.0$, we have *internal reflection*, and the reflection coefficient for this case is shown in figure I.10. For internal angles of incidence less than about 35°, little light is reflected, the rest being transmitted. After this, the reflection coefficient increases rapidly, until the internal angle of incidence θ_1 approaches a *critical angle C*, given by

$$\sin C = \frac{n_2}{n_1}, \qquad 1.5.3$$

This corresponds to an angle of emergence of 90°. For angles of incidence greater than this, the light is *totally internally reflected.* For glass of refractive index 1.5, the critical angle is 41°.2, so that light is totally internally reflected inside a 45° prism such as is used in binoculars.



1.6 *Refraction by a Prism*

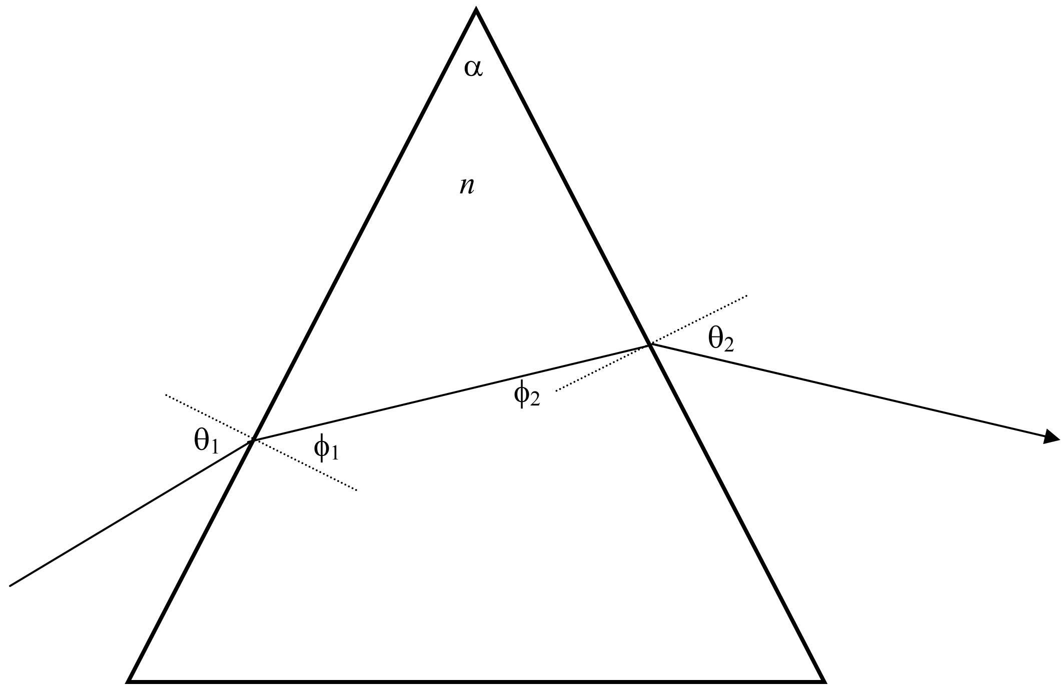


FIGURE I.11

Figure I.11 shows an isosceles prism of angle α, and a ray of light passing through it. I have drawn just one ray of a single colour. For white light, the colours will be *dispersed*, the violet light being deviated by the prism more than the red light. We'll choose a wavelength such that the refractive index of the prism is n. The *deviation D* of the light from its original direction is $\theta_1 - \phi_1 + \theta_2 - \phi_2$. I want to imagine, now, if we keep the incident ray fixed and rotate the prism, how does the deviation vary with angle of incidence θ_1? By geometry, $\phi_2 = \alpha - \phi_1$, so that the deviation is

$$D = \theta_1 + \theta_2 - \alpha. \qquad 1.6.1$$

Apply Snell's law at each of the two refracting surfaces:

$$\frac{\sin\theta_1}{\sin\phi_1} = n \quad \text{and} \quad \frac{\sin\theta_2}{\sin(\alpha - \phi_1)} = n\ , \qquad 1.6.2\text{a,b}$$

and eliminate ϕ_1:

$$\sin\theta_2 = \sin\alpha\sqrt{n^2 - \sin^2\theta_1} - \cos\alpha\sin\theta_1. \qquad 1.6.3.$$

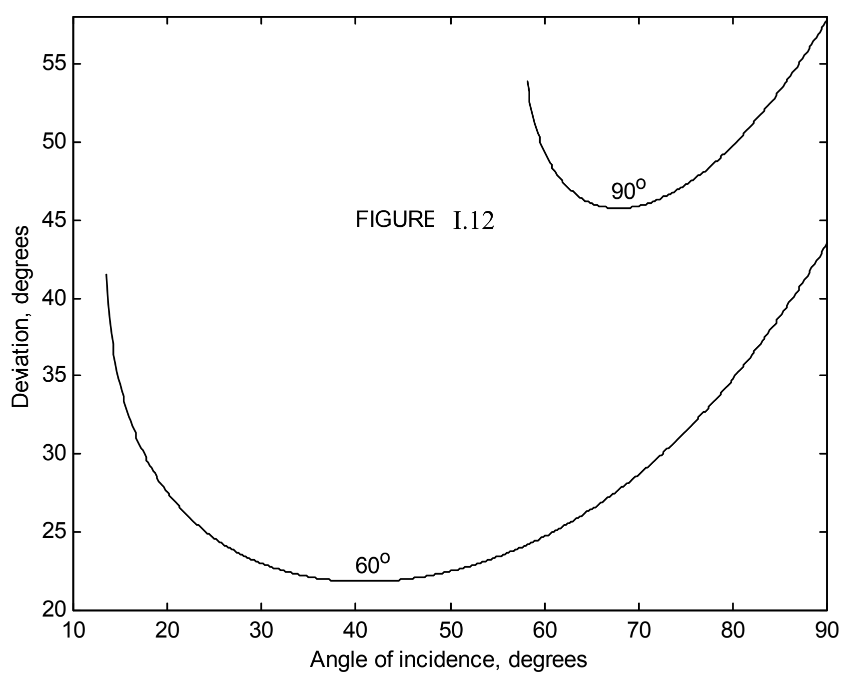
Equations 1.6.1 and 1.6.3 enable us to calculate the deviation as a function of the angle of incidence θ_1. The deviation is least when the light traverses the prism symmetrically, with $\theta_1 = \theta_2$, the light inside the prism then being parallel to the base. Putting $\theta_1 = \theta_2$ in equation shows that minimum deviation occurs for an angle of incidence given by

$$\sin\theta_1 = \frac{n\sin\alpha}{\sqrt{2(1+\cos\alpha)}} = n\sin\tfrac{1}{2}\alpha\ . \qquad 1.6.4$$

The angle of minimum deviation $D_{\min}$ is $2\theta_1 - \alpha$, where θ_1 is given by equation 1.6.4, and this leads to the following relation between the refractive index and the angle of minimum deviation:

$$n = \frac{\sin\frac{1}{2}(D_{\min} + \alpha)}{\sin\frac{1}{2}\alpha}. \qquad 1.6.5$$

Of particular interest are prisms with $\alpha = 60^\circ$ and $\alpha = 90^\circ$. I have drawn, in figure I.12 the deviation versus angle of incidence for 60- and 90-degree prisms, using (for reasons I shall explain) $n = 1.31$, which is approximately the refractive index of ice. For the 60° ice prism, the angle of minimum deviation is $21^\circ.8$, and for the 90° ice prism it is $45^\circ.7$.



The geometry of refraction by a regular hexagonal prism is similar to refraction by an equilateral (60^o) triangular prism (figure I.13):

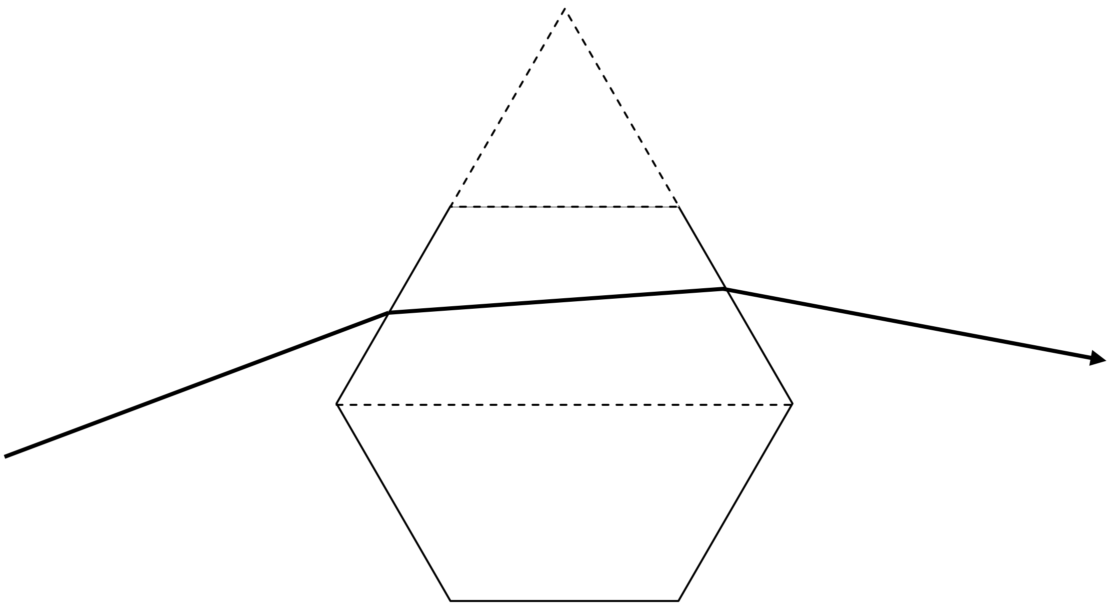


FIGURE I.13

When hexagonal ice crystals are present in the atmosphere, sunlight is scattered in all directions, according to the angles of incidence on the various ice crystals (which may or may not be oriented randomly). However, the rate of change of the deviation with angle of incidence is least near minimum deviation; consequently much more light is deviated by $21^o.8$ than through other angles. Consequently we see a halo of radius about 22^o around the Sun.

Seen sideways on, a hexagonal crystal is rectangular, and consequently refraction is as if through a 90^o prism (figure I.14):

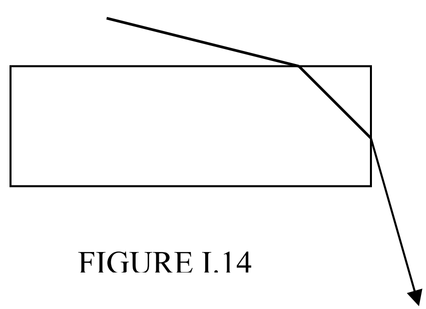


FIGURE I.14

Again, the rate of change of deviation with angle of incidence is least near minimum deviation, and consequently we may see another halo, or radius about 46^o. For both haloes, the violet is deviated more than the red, and therefore both haloes are tinged violet on the outside and red on the inside.

1.7 *The Rainbow*

I do not know the exact shape of a raindrop, but I doubt very much if it is drop-shaped! Most raindrops will be more or less spherical, especially small drops, because of surface tension. If large, falling drops are distorted from an exact spherical shape, I imagine that they are more likely to be flattened to a sort of horizontal pancake shape rather than drop-shaped. Regardless, in the analysis in this section, I shall assume drops are spherical, as I am sure small drops will be.

We wish to follow a light ray as it enters a spherical drop, is internally reflected, and finally emerges. See figure I.15.

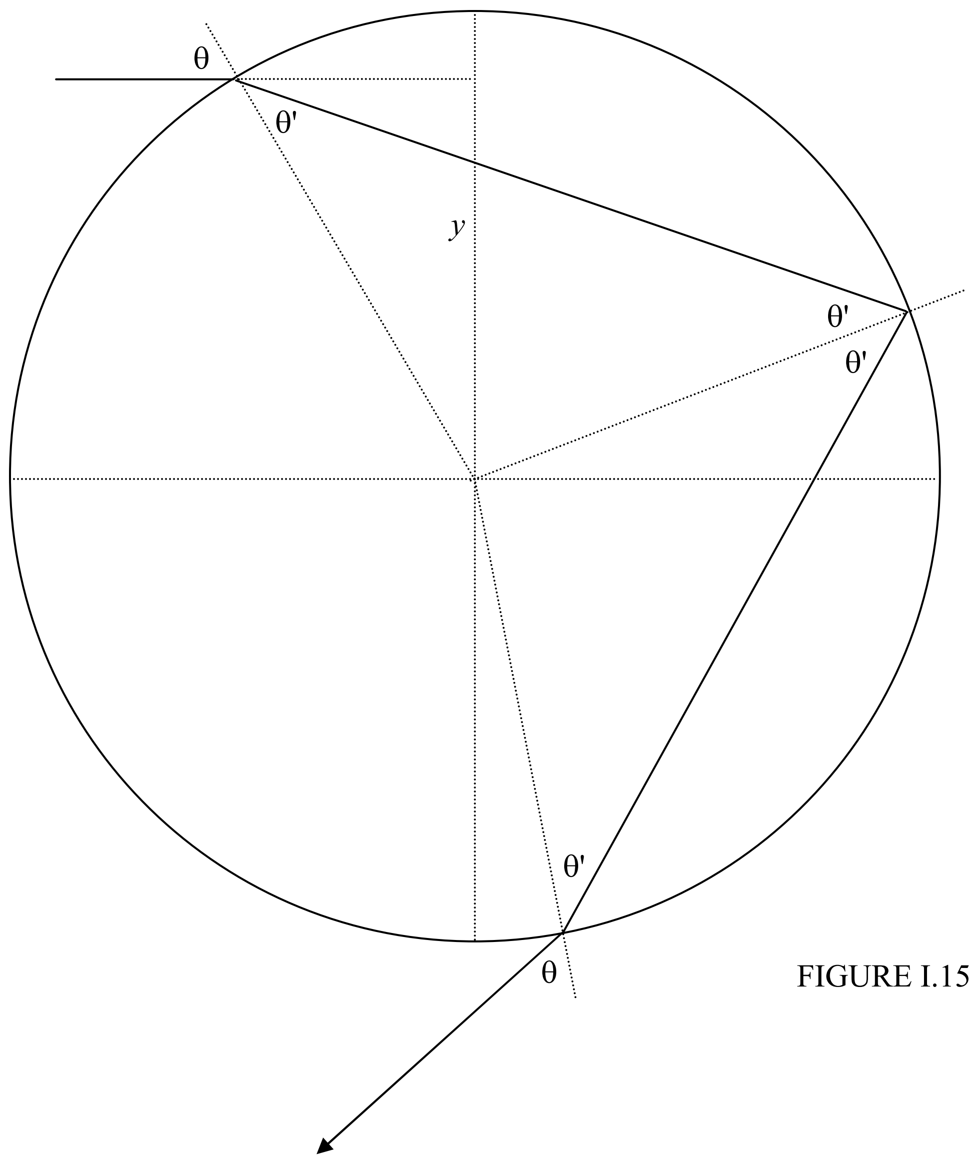


FIGURE I.15

We see a ray of light headed for the drop, which I take to have unit radius, at impact parameter y. The deviation of the direction of the emergent ray from the direction of the incident ray is

$$D = \theta - \theta' + \pi - 2\theta' + \theta - \theta' = \pi + 2\theta - 4\theta'. \qquad 1.7.1$$

However, we shall be more interested in the angle $r = \pi - D$. A ray of light that has been deviated by D will approach the observer from a direction that makes an angle r from the centre of the rainbow, which is at the anti-solar point (figure I.16):

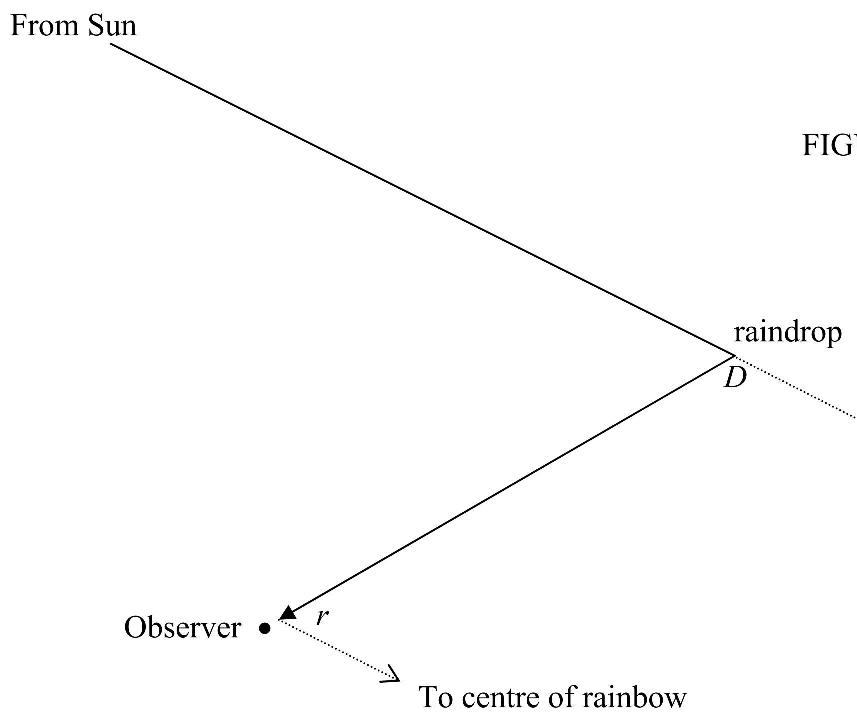


FIGURE I.16

We would like to find the deviation D as a function of impact parameter. The angles of incidence and refraction are related to the impact parameter as follows:

$$\sin\theta = y, \qquad 1.7.2$$

$$\cos\theta = \sqrt{1 - y^2}, \qquad 1.7.3$$

$$\sin\theta' = y/n, \qquad 1.7.4$$

and

$$\cos\theta = \sqrt{1 - y^2/n^2}. \qquad 1.7.5$$

These, together with equation 1.7.1, give us the deviation as a function of impact parameter. The deviation goes through a minimum – and r goes through a maximum. The deviation for a light ray of impact parameter y is

$$D = \pi + 2\sin^{-1} y - 4\sin^{-1}(y/n). \qquad 1.7.6$$

This is shown in figure I.17 for $n = 1.3439$ (blue - $\lambda = 400$ nm) and $n = 1.3316$ (red - $\lambda = 650$ nm).

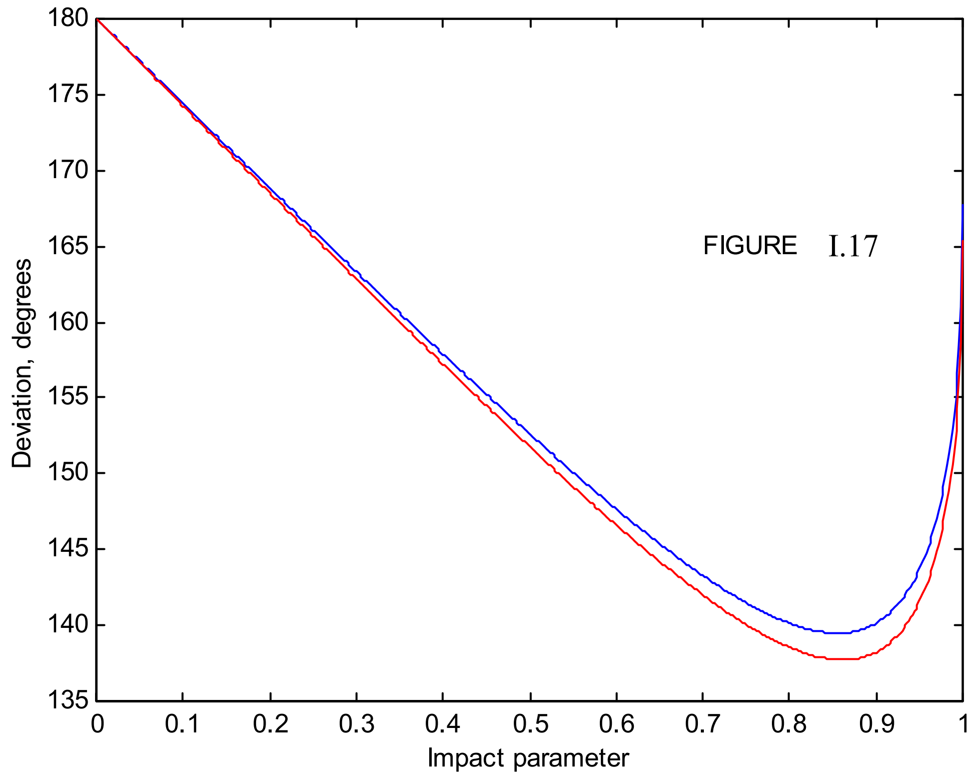


FIGURE I.17

The angular distance r from the centre of the bow is $r = \pi - D$, so that

$$r = 4\sin^{-1}(y/n) - 2\sin^{-1} y. \qquad 1.7.7$$

Differentiation gives the maximum value, R, of r - i.e. the radius of the bow – or the minimum deviation $D_{\min}$. We obtain for the radius of the bow

$$R = 4\sin^{-1}\sqrt{\frac{4 - n^2}{3n^2}} - 2\sin^{-1}\sqrt{\frac{4 - n^2}{3}}. \qquad 1.7.8$$

For $n = 1.3439$ (blue) this is 40° 31' and for $n = 1.3316$ (red) this is 42° 17'. Thus the blue is on the inside of the bow, and red on the outside.

For grazing incidence (impact parameter = 1), the deviation is $2\pi - 4\sin^{-1}(1/n)$, or 167° 40' for blue or 165° 18' for red. This corresponds to a distance from the center of the bow $r = 4\sin^{-1}(1/n) - \pi$, which is 12° 20' for blue and 14° 42' for red. It will be seen from figure I.17 that for deviations between D_{min} and about 166^o there are *two* impact parameters that result in the same deviation. The paths of two rays with the same deviation are shown in figure I.18. One ray is drawn as a full line, the other as a dashed line. They start with different impact parameters, and take different paths through the drop, but finish in the same direction. The drawing is done for a deviation of 145^o, or 35^o from the bow centre. The two impact parameters are 0.969 and 0.636. When these two rays are recombined by being brought to a focus on the retina of the eye, they have satisfied all the conditions for interference, and the result will be brightness or darkness according as to whether the path difference is an even or an odd number of half wavelengths.

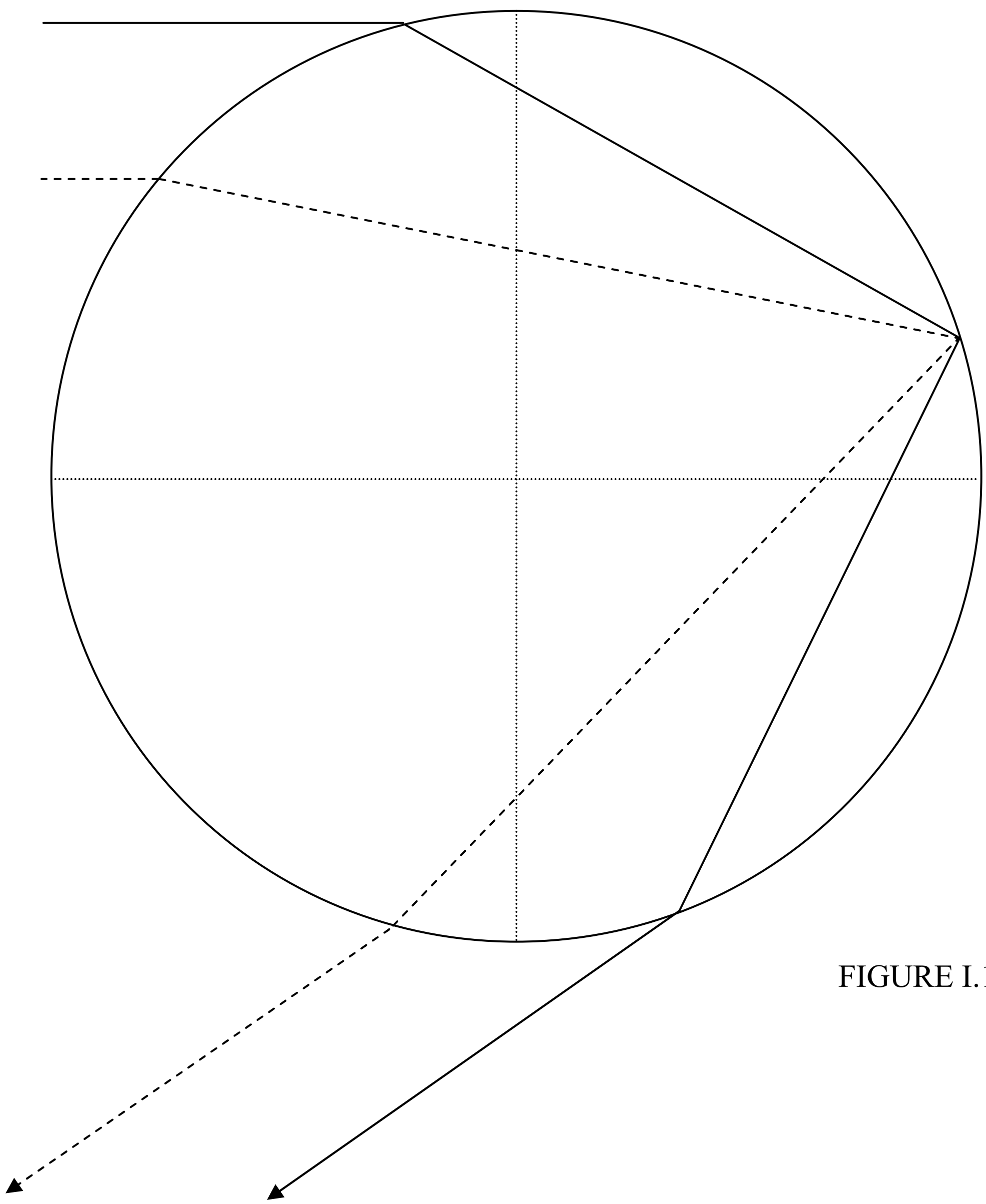


FIGURE I.18

If you look just inside the inner (blue) margin of the bow, you can often clearly see the interference fringes produced by two rays with the same deviation. I haven't tried, but if you were to look through a filter that transmits just one colour, these fringes (if they are bright enough to see) should be well defined. The optical path difference for a given deviation, or given *r*, depends on the radius of the drop (and on its refractive index). For a drop of radius *a* it is easy to see that the optical path difference is

$$2a(\cos\theta_2 - \cos\theta_1) - 4n(\cos\theta'_2 - \cos\theta'_1),$$

where θ_1 is the larger of the two angles of incidence. Presumably if you were to measure the fringe spacing, you could determine the size of the drops. Or, if you were to conduct a Fourier analysis of the visibility of the fringes, you could determine, at least in principle, the size distribution of the drops.

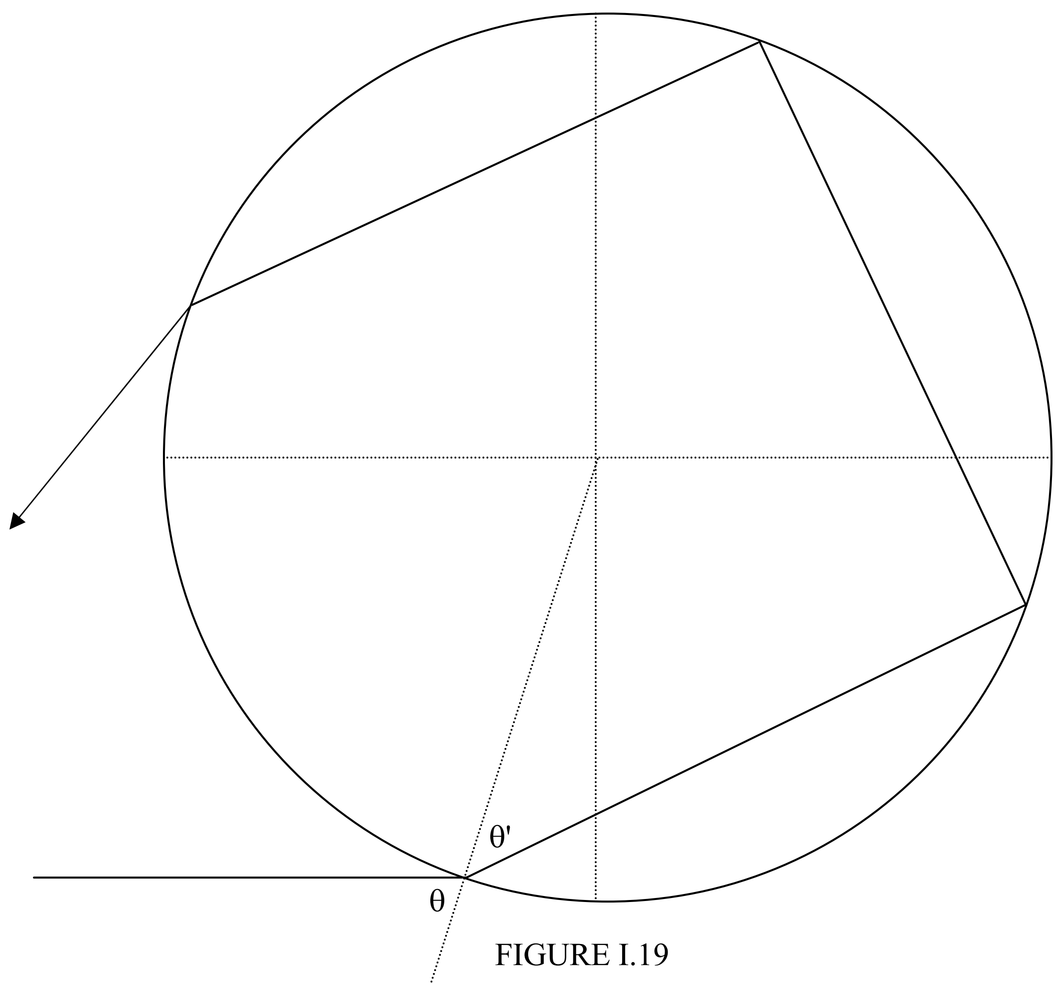


FIGURE I.19

Some distance outside the primary rainbow, there is a secondary rainbow, with colours reversed – i.e. red on the inside, blue on the outside. This is formed by two internal reflections inside the drop (figure I.19). The deviation of the final emergent ray from the direction of the incident ray is $(\theta - \theta') + (\pi - 2\theta') + (\pi - 2\theta') + (\theta - \theta')$, or $2\pi + 2\theta - 6\theta'$ counterclockwise, which amounts to $D = 6\theta' - 2\theta$ clockwise. That is,

$$D = 6\sin^{-1}(y/n) - 2\sin^{-1} y. \qquad 1.7.9$$

clockwise, and, as before, this corresponds to an angular distance from the centre of the bow $r = \pi - D$. I show in figure I.20 the deviation as a function of impact parameter y. Notice that D goes through a *maximum* (and hence r has a *minimum* value). There is no light scattered outside the primary bow, and no light scattered inside the secondary bow. When the full glory of a primary bow and a secondary bow is observed, it will be seen that the space between the two bows is relatively dark, whereas it is brighter inside the primary bow and outside the secondary bow.

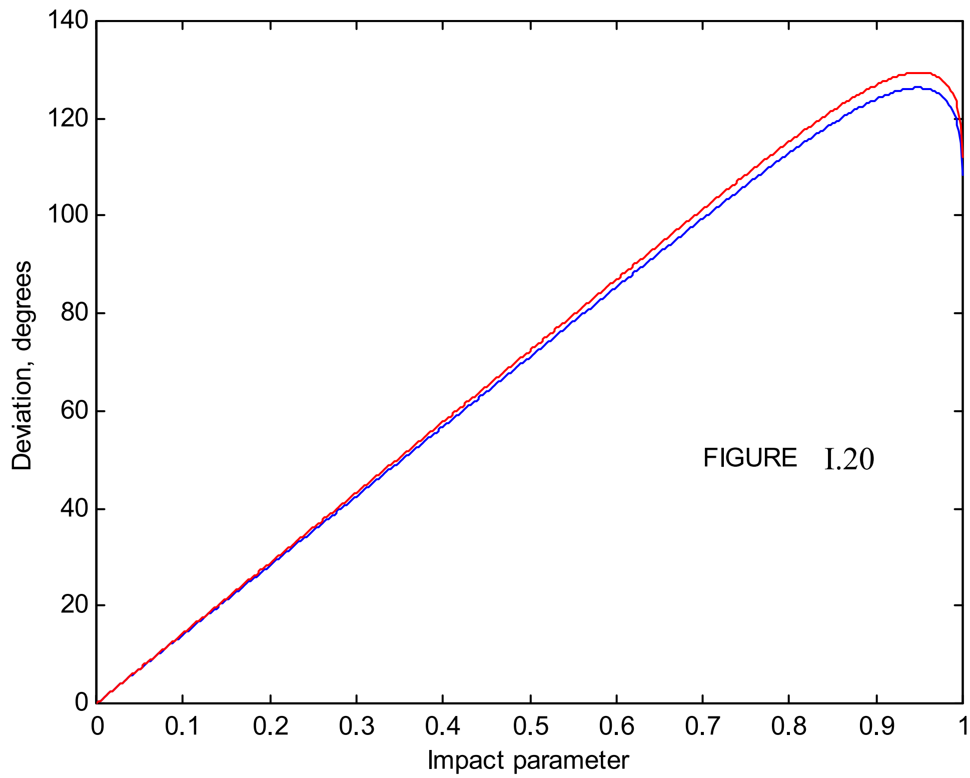


FIGURE I.20

Differentiation shows that the least value of r, (greatest deviation) corresponding to the radius of the secondary bow is

$$R = 6\sin^{-1}\sqrt{\frac{3 - n^2}{2n^2}} - 2\sin^{-1}\sqrt{\frac{3 - n^2}{2}} . \qquad 1.7.10$$

For $n = 1.3439$ (blue) this is 53° 42' and for $n = 1.3316$ (red) this is 50° 31'. Thus the red is on the inside of the bow, and blue on the outside.

Problem. In principle a tertiary bow is possible, involving three internal reflections. I don't know if anyone has observed a tertiary bow, but I am told that the primary bow is blue on the inside, the secondary bow is red on the inside, and "therefore" the tertiary bow would be blue on the inside. On the contrary, I assert that the tertiary bow would be red on the inside. Why is this?

Let us return to the primary bow. The deviation is (equation 1.7.1) $D = \pi + 2\theta - 4\theta'$. Let's take $n = 4/3$, which it will be for somewhere in the middle of the spectrum. According to equation 1.7.8, the radius of the bow ($R = \pi - D_{\rm min}$) is then about 42°. That is, $2\theta' - \theta = 21^\circ$. If we combine this with Snell's law, $3\sin\theta = 4\sin\theta'$, we find that, at minimum deviation (i.e. where the primary bow is), $\theta = 60^\circ.5$ and $\theta' = 40^\circ.8$. Now, at the point of internal reflection, not all of the light is reflected (because θ' is less than the critical angle of $36^\circ.9$), and it will be seen that the angle between the reflected ands refracted rays is $(180 - 60.6 - 40.8)$ degrees $= 78^\circ.6$. Those readers who are familiar with Brewster's law will understand that when the reflected and transmitted rays are at right angles to each other, the reflected ray is completely plane polarized. The angle, as we have seen, is not 90°, but is $78^\circ.6$, but this is sufficiently close to the Brewster condition that the reflected light, while not completely plane polarized, is strongly polarized. Thus, as can be verified with a polarizing filter, the rainbow is strongly plane polarized.

I now want to address the question as to how the brightness of the bow varies from centre to circumference. It is brightest where the slope of the deviation versus impact parameter curve is least – i.e. at minimum deviation (for the primary bow) or maximum deviation (for the secondary bow). Indeed the radiance (surface brightness) at a given distance from the centre of the bow is (among other things) inversely proportional to the slope of that curve. The situation is complicated a little in that, for deviations between $D_{\rm min}$ and $2\pi - 4\sin^{-1}(1/n)$, (this latter being the deviation for grazing incidence), there are two impact parameters giving rise to the same deviation, but for deviations greater than that (i.e. closer to the centre of the bow) only one impact parameter corresponds to a given deviation.

Let us ask ourselves, for example, how bright is the bow at 35° from the centre (deviation 145°)? The deviation is related to impact parameter by equation 1.7.6. For $n = 4/3$, we find that the impact parameters for deviations of 144, 145 and 146 degrees are as follows:

D°	y
144	0.6583 and 0.9623
145	0.6366 and 0.9693
146	0.6157 and 0.9736

Figure I.21 shows a raindrop seen from the direction of the approaching photons.

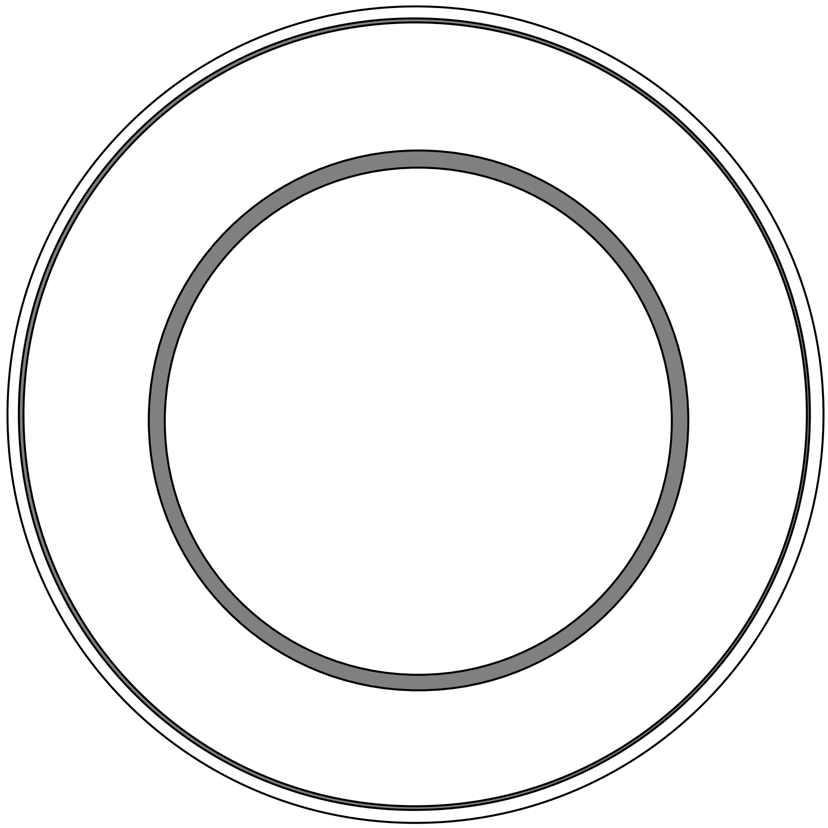
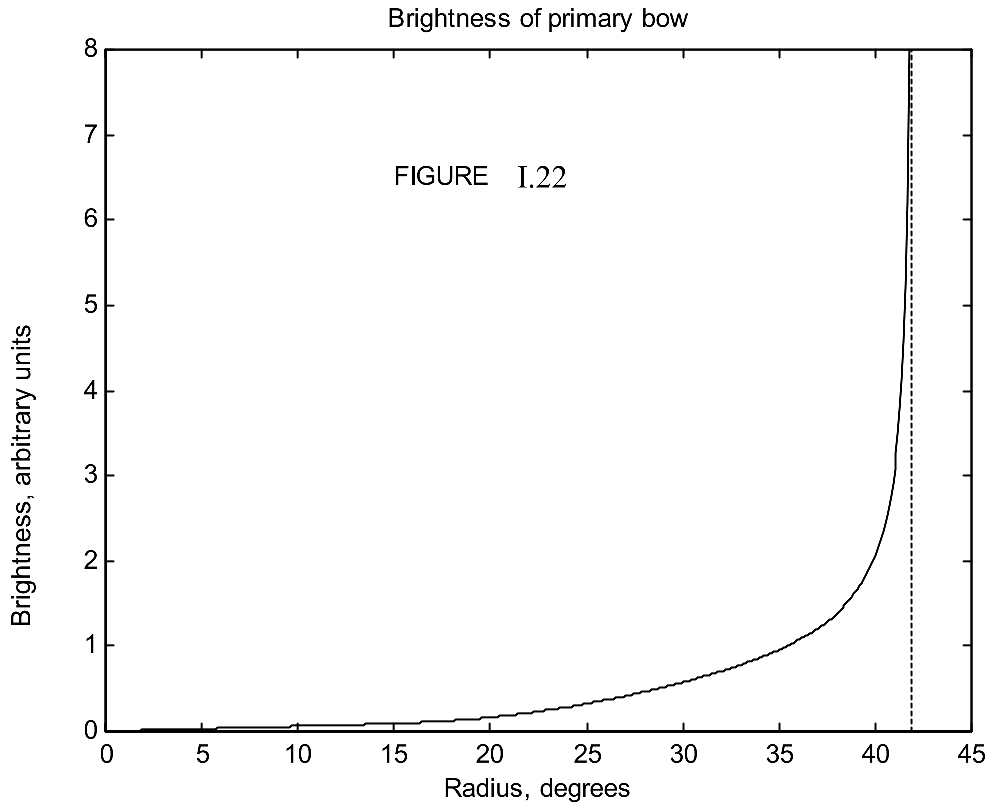


FIGURE I.21

Any photons with impact parameters within the two dark annuli will be deviated between 144° and 146°, and will ultimately approach the observer at angular distances between 36° and 34° from the centre. The radiance at a distance of 35° from the centre will be proportional, among other things, to the sum of the areas of these two annuli.

I have said “among other things”. Let us now think about other things. I have drawn figure I.15 as if all of the light is transmitted as it enters the drop, and then all of it is internally reflected within the drop, and finally all of it emerges when it leaves the drop. This is not so, of course. At entrance, at internal reflection and at emergence, some of the light is reflected and some is transmitted. The fractions that are reflected or transmitted depend on the angle of incidence, but, for minimum deviation, about 94% is transmitted on entry to and again at exit from the drop, but only about 6% is internally reflected. Also, after entry, internal reflection and exit, the percentage of polarization of the ray increases. The formulas for the reflection and transmission coefficients (Fresnel’s equations) are somewhat complicated (equations 1.5.1 and 1.5.2 are for unpolarized incident light), but I have followed them through as a function of impact parameter, and have also taken account of the sizes of the one or two annuli involved for each impact parameter, and I have consequently calculated the variation of surface brightness for one colour ($n = 4/3$) from the centre to the circumference of the bow. I omit the details of the calculations, since this chapter was originally planned as an elementary account of reflection and transmission, and we seem to have gone a little beyond that, but I show the results of the calculation in figure I.22. I have not, however, taken account of the interference phenomena, which can often be clearly seen just within the primary bow.



1.8 *Problem*

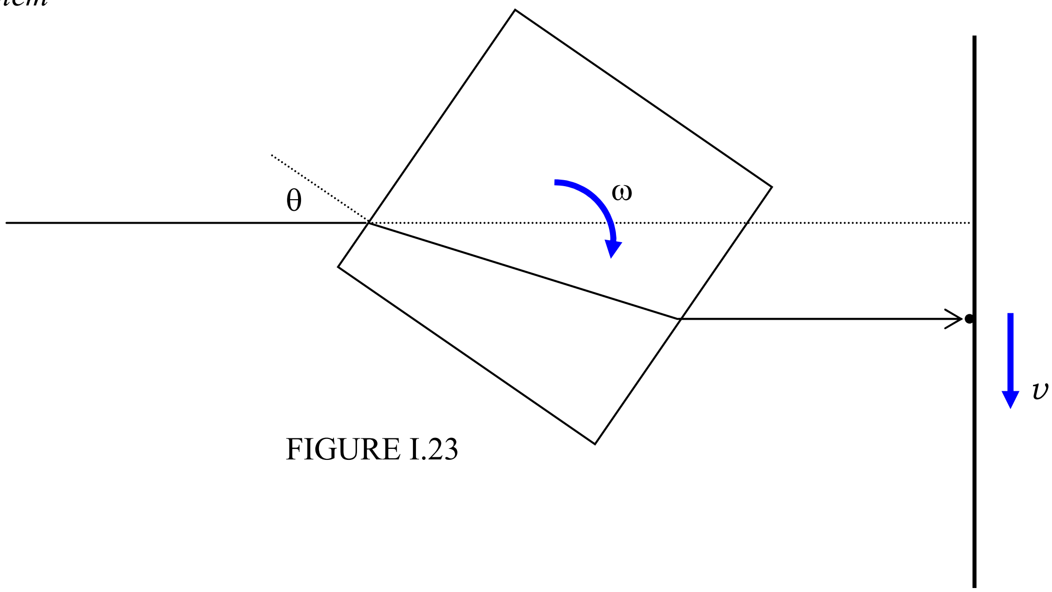


FIGURE I.23

See figure I.23. A ray of light is directed at a glass cube of side a, refractive index n, eventually to form a spot on a screen beyond the cube. The cube is rotating at an angular speed ω. Show that, when the angle of incidence is θ, the speed of the spot on the screen is

$$\upsilon = a\omega\left(\cos\theta - \frac{n^2\cos 2\theta + \sin^4\theta}{(n^2 - \sin^2\theta)^{3/2}}\right)$$

and that the greatest displacement of the spot on the screen from the undisplaced ray is

$$D = \frac{a}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{n^2 - \sin^2\theta}}\right).$$

I refrain from asking what is the maximum speed and for what value of θ does it occur. However, I ran the equation for the speed on the computer, with $n = 1.5$, and, if the formula is right, the speed is $\frac{1}{3}a\omega$ when $\theta = 0$, and it increases monotonically up to $\theta = 45^{\circ}$, which is as far as we can go for a cube. However, if we have a rectangular glass block, we can increase θ to 90°, at which time the speed is 0.8944 $a\omega$. The speed goes through a maximum of about $0.9843a\omega$ when $\theta = 79^{\circ}.3$. I'd be interested if anyone can confirm this, and do it analytically.

1.9 *Differential Form of Snell's Law*

Snell's law in the form $n \sin\theta$ = constant is useful in calculating how a light ray is bent in travelling from one medium to another where there is a discrete change of refractive index. If there is a medium in which the refractive index is changing continuously, a differential form of Snell's law may be useful. This is obtained simply by differentiation of $n \sin\theta$ = constant, to obtain the differential form of Snell's law:

$$\cot\theta\, d\theta \ = \ -\frac{dn}{n}. \qquad 1.9.1$$

Let us see how this might be used. Let us suppose, for example, that we have some medium in which the refractive index diminishes with height y according to

$$n \ = \frac{a}{a - y}. \qquad 1.9.2$$

Here a is an arbitrary distance, and I am going to restrict our interest only to heights less than a – so that n doesn't become infinite! I have chosen equation 1.9.2 only because it happens to lead to a rather simple result. Let us suppose that we direct a light ray upwards from the origin in a direction making an angle α with the horizontal, and we wish to trace the ray through the medium as the refractive index continuously changes. See figure I.24.

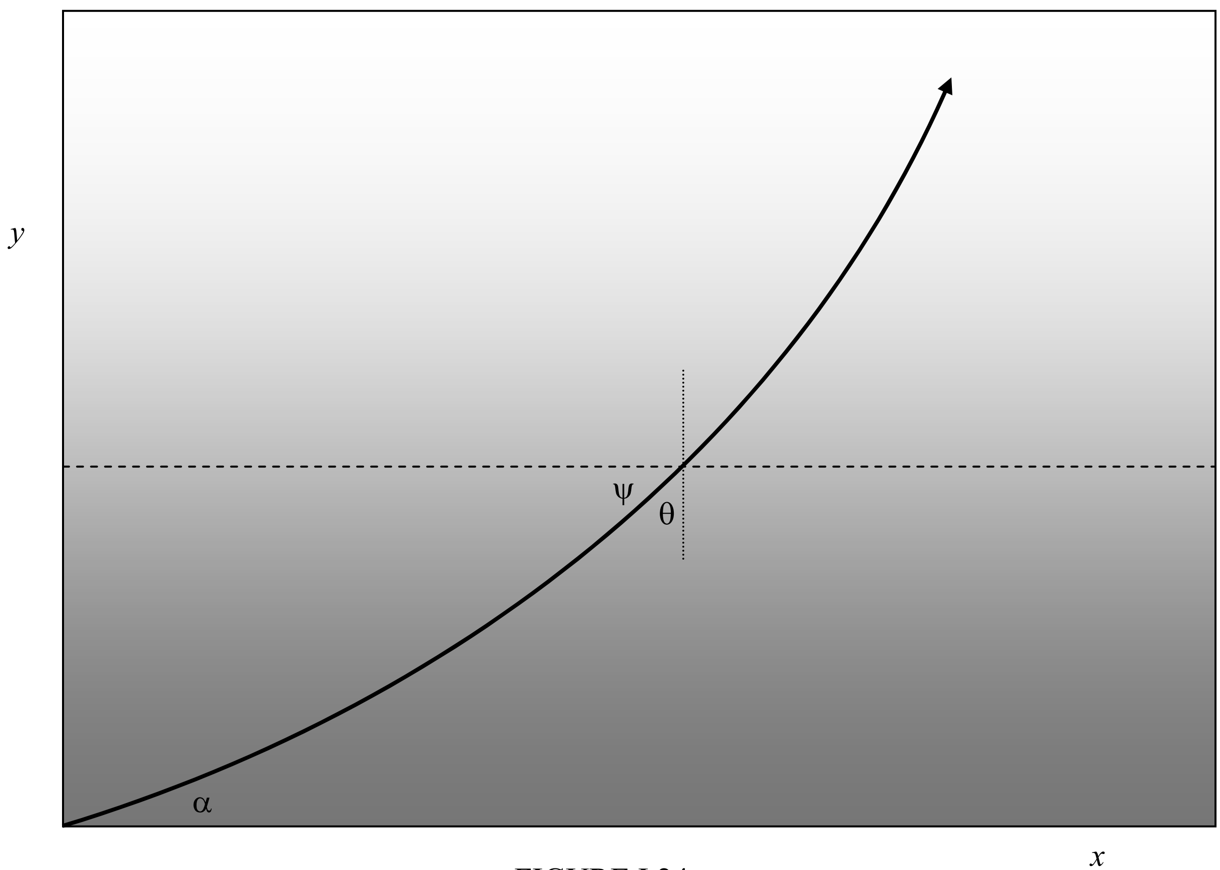


FIGURE I.24

When the height is y, the angle of incidence is θ, and the slope $dy/dx = \tan\psi$, where $\psi = \pi/2 - \theta$. With this and equation 1.9.2, Snell's law takes the form

$$\tan\psi \; d\psi \;=\; \frac{dy}{a-y}. \qquad 1.9.3$$

On integration, this becomes

$$(a - y)\sec\psi \;=\; \text{constant} \;=\; a\sec\alpha. \qquad 1.9.4$$

Let $a - y = \eta$ and $a\sec\alpha = c$. Equation 1.9.4 then becomes

$$\sec\psi \;=\; c/\eta. \qquad 1.9.5$$

But $\tan\psi \;=\; \sqrt{\sec^2\psi - 1} \;=\; dy/dx \;=\; -d\eta/dx$, so we obtain

$$\frac{d\eta}{dx} \;=\; -\frac{\sqrt{c^2 - \eta^2}}{\eta}. \qquad 1.9.6$$

On integration, this becomes

$$x \;=\; \sqrt{c^2 - \eta^2} \;+\; C. \qquad 1.9.7$$

We recall that $a - y = \eta$ and $a\sec\alpha = c$, from which equation 1.9.7 becomes

$$x \;=\; \sqrt{a^2\tan^2\alpha \;+\; 2ay \;-\; y^2} \;+\; C. \qquad 1.9.8$$

Since the ray starts at the origin, it follows that $C \;=\; -a\tan\alpha$. The path of the ray, therefore, is found, after some algebra, to be

$$(x + a\tan\alpha)^2 \;+\; (y - a)^2 \;=\; a^2\sec^2\alpha, \qquad 1.9.9$$

which is a circle, centre $(-a\tan\alpha,\, a)$, radius $a\sec\alpha$.